Coinductive Predicates and Final Sequences in a Fibration\textsuperscript{†}

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Received 8 January 2015

Coinductive predicates express persisting “safety” specifications of transition systems. Previous observations by Hermida and Jacobs identify coinductive predicates as suitable final coalgebras in a fibration—a categorical abstraction of predicate logic. In this paper we follow the spirit of a seminal work by Worrell and study final sequences in a fibration. Our main contribution is to identify some categorical “size restriction” axioms that guarantee stabilization of final sequences after $\omega$ steps. In its course we develop a relevant categorical infrastructure that relates fibrations and locally presentable categories, a combination that does not seem to be studied a lot. The genericity of our fibrational framework can be exploited for: binary relations (i.e. the logic of “binary predicates”) for which a coinductive predicate is bisimilarity; constructive logics (where interests are growing in coinductive predicates); and logics for name-passing processes.

1. Introduction

Coinductive predicates postulate properties of state-based dynamic systems that persist after a succession of transitions. In computer science, safety properties of nonterminating, reactive systems are examples of paramount importance. This has led to an extensive study of specification languages in the form of fixed point logics and model-checking algorithms.

In this paper we follow (Hermida and Jacobs, 1998; Hermida, 1993)—whose results are further extended in (Fumex et al., 2011; Atkey et al., 2012), see also (Jacobs, 2012, Chap. 6)—and take a categorical view on coinductive predicates. Here coalgebras represent transition systems; a fibration is a “predicate logic”; and a coinductive predicate is

\textsuperscript{†} An earlier version of this paper (Hasuo et al., 2013) has been presented at Mathematical Foundations of Programming Semantics, Twenty-Ninth Annual Conference (MFPS XXIX), 23-25 June 2013, Tulane University, New Orleans, Louisiana, USA.

\textsuperscript{‡} The main part of this work was done when K.C. was an MSc student at Department of Computer Science, the University of Tokyo.
identified as a suitable coalgebra in a fibration. Our contribution is the study of final sequences—an iterative construction of final coalgebras that is studied notably in (Worrell, 2005; Adámek, 2003)—in such a fibrational setting.

Coalgebras have been successfully used as a categorical abstraction of transition systems (see e.g. (Rutten, 2000; Jacobs, 2012)): by varying base categories and functors, coalgebras bring general results that work for a variety of systems at once. Fixed point logics (or modal logics in general), too, have been actively studied coalgebraically: coalgebraic modal logic is a prolific research field (see (Cîrstea et al., 2011)); their base category is typically Sets but works like (Klin, 2007) go beyond and use presheaf categories for processes in name-passing calculi; and literature including (Cîrstea and Sadrzadeh, 2008; Venema, 2006; Cîrstea et al., 2009) studies coalgebraic fixed point logics.

Unlike most of these works, we follow (Hermida and Jacobs, 1998; Hermida, 1993) and parametrize the underlying “predicate logic” too with the categorical notion of fibration. The conventional setting of classical logic is represented by the fibration (see Appendix C for an introduction to fibrations).

\[ \text{fibration } \mathcal{E}_{\mathcal{C}} \downarrow \mathcal{P} \]

\[ \text{coalgebra } \text{invariant bisimulation} \]

\[ \begin{array}{ccc}
\text{final coalgebra} & \text{coinductive predicate} & \text{bisimilarity} \\
\text{coinductive predicates} & \text{bisimilarity} & \text{relations} \\
\end{array} \]

However there are various other “logics” modeled as fibrations, and hence the fibrational language provides a uniform treatment of these different settings. An example is binary relations (instead of unary predicates) that form a fibration (see Appendix C). In this case coinductive predicates are bisimilarity relations (see the above table, and Example 7.2 later).

Another example is predicates in constructive logics. They are modeled by the subobject fibration of a topos. In fact, coinductive predicates in constructive logics are an emerging research topic: coinduction is supported in the theorem prover Coq (based on the constructive calculus of constructions), see e.g. (Bertot and Komendantskaya, 2008); and, working in Coq, some interesting differences between classically equivalent (co)inductive predicates have been studied e.g. in (Nakata et al., 2011).

Yet another example is modal logics for processes in various name-passing calculi. They are best modeled by the subobject fibration of a suitable (pre)sheaf category like Sets and Sets\(^F\) (Fiore and Turi, 2001; Fiore and Staton, 2006; Miculan, 2008; Staton, 2011).

1.1. Coinductive Predicates and Their Construction, Conventionally

In order to illustrate our technical contributions (§3) we here present a special case, with classical logic and Kripke models. We first introduce syntax.

**Definition 1.1 (Rudimentary logic \(R\nu\)).** In this tiny fragment of the \(\mu\)-calculus, fixed-point operators are limited to the greatest one at the outermost position; and moreover
all the formulas are “rank-1,” that is, the fixed-point variable $u$ occurs precisely under one modal operator.

$$
Rv \ni \alpha ::= a \mid \pi \mid \Box u \mid \lozenge u \mid \alpha \land \alpha \mid \alpha \lor \alpha ; \quad Rv \ni \beta ::= \nu u.\alpha .
$$

(1)

Here $a$ belongs to the set $\text{AP}$ of atomic propositions; $\pi$ stands for the negation of $a$; and $u$ is the only fixed-point variable (with possibly multiple occurrences).

An $Rv$-formula can be thought of as a recursive definition of a coinductive predicate. Later we will model such a “definition” categorically as a predicate lifting. Among specifications expressible in $Rv$ is (may-) deadlock freedom (“there is an infinite path”). It is expressed by $\nu u.\lozenge u$ and is our recurring example.

An $Rv$-formula is interpreted in Kripke models. Let $c = (X, \rightarrow, V)$ be a Kripke model, where $X$ is a state space, $\rightarrow \subseteq X \times X$ is a transition relation and $V : X \rightarrow \mathcal{P}(\text{AP})$ is a valuation. The conventional interpretation $[\nu u.\alpha]_c$ of $Rv$-formulas in the Kripke model $c$ is given as follows (see e.g. (Bradfield and Stirling, 2006)). Firstly, we interpret $\alpha \in Rv_u$ as a function $[\alpha]_c : \mathcal{P}X \rightarrow \mathcal{P}X$. Concretely:

$$
[a]_c(P) = \{x \mid a \in V(x)\} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad "
Besides a state space bound, another (possibly better and seemingly less known) bound can be obtained from a behavioral view. One realizes that not only the size of the state space \( X \) but also the branching degree can be used to bound the length of the chain (2). This is a result similar to the one in (Hennessy and Milner, 1985, Thm. 2.1); the latter is stated for bisimilarity as a coinductive relation, not for a coinductive predicate. We formally state (an instance of) the result for the record.

**Lemma 1.2 (Behavioral bound).** Let \( c = (X, \rightarrow, V) \) be a finitely branching Kripke model. For \( \alpha = \Diamond u \), the chain (2) stabilizes after \( \omega \) steps and yields \( [\nu u. \Diamond u]_c \) as its limit, that is, \( \Lambda_{i \in \omega} ([\Diamond u]_c^i \top) = [\nu u. \Diamond u]_c \).

*Proof.* The essence of the result lies in the fact that the limit \( \Lambda_{i \in \omega} ([\Diamond u]_c^i \top) \) is a \( \Diamond \)-invariant, which we shall prove now. Assume that a state \( x \) satisfies \( \Lambda_{i \in \omega} ([\Diamond u]_c^i \top) \); we have to show that \( x \) satisfies \( [\Diamond u]_c \left( \Lambda_{i \in \omega} ([\Diamond u]_c^i \top) \right) \), that is, there is a successor \( x' \) of \( x \) that satisfies the limit \( \Lambda_{i \in \omega} ([\Diamond u]_c^i \top) \).

Since \( x \) satisfies \( [\Diamond u]_c^i \top \) ("there is a path of length \( \geq i \)) for each \( i \), for each \( i \geq 1 \), there is a successor \( x_i \) of \( x \) that satisfies \( [\Diamond u]_c^{i-1} \top \). By \( c \) being finitely branching, the set \( \{x_1, x_2, \ldots \} \) of such successors turns out to be finite and there exists a successor \( x' \) of \( x \) such that \( x' = x_i \) for infinitely many \( i \). It follows (from \( [\Diamond u]_c^j \top \leq [\Diamond u]_c^i \top \) if \( j \leq i \)) that this \( x' \) satisfies \( [\Diamond u]_c^i \top \) for all \( i \in \omega \), and hence satisfies \( \Lambda_{i \in \omega} ([\Diamond u]_c^i \top) \). This proves that the limit \( \Lambda_{i \in \omega} ([\Diamond u]_c^i \top) \) is an invariant, and hence \( \Lambda_{i \in \omega} ([\Diamond u]_c^i \top) \leq [\nu u. \Diamond u]_c \).

For the last equality claimed in the lemma, the other direction \( [\nu u. \Diamond u]_c \leq \Lambda_{i \in \omega} ([\Diamond u]_c^i \top) \) is easy: \([\nu u. \Diamond u]_c \leq [\Diamond u]_c^i \top\) is easily shown by induction on \( i \). This concludes the proof. \( \square \)

Note that Lem. 1.2 holds however large the state space \( X \) is. Moreover it easily generalizes from \( \nu u. \Diamond u \rightarrow \) to an arbitrary \( \nu u. \alpha \). Note also that the counterexample \( c_1 \) in (3) is not finitely branching and does not contradict with Lem. 1.2.

### 1.2. Final Sequences in a Fibration

This paper is about putting the observations in §1.1 in general categorical terms. Our starting observation is that the chain (2) resembles a final sequence, a classic construction of a final coalgebra.

In the theory of coalgebra a final \( F \)-coalgebra is of prominent importance since it is a fully abstract domain with respect to the \( F \)-behavioral equivalence. Therefore a natural question is if a final \( F \)-coalgebra exists; the well-known Lambek lemma prohibits e.g. a final \( \mathcal{P} \)-coalgebra for the (full) powerset functor \( \mathcal{P} \). What matters is the size of \( F \): when it is suitably bounded, it is known that a final coalgebra can be constructed via the following final \( F \)-sequence.

\[
1 \leftarrow F \leftarrow \cdot \cdot \cdot \leftarrow F^{i+1} 1 \leftarrow F^{i+1} \cdot \cdot \cdot \tag{4}
\]

Here 1 is a final object in \( \mathcal{C} \), and \( ! \) is the unique arrow. In particular, if \( F \) is finitary, a final coalgebra arises as a suitable subobject (or a quotient) of the \( \omega \)-limit of the final sequence (4). These constructions in \textbf{Sets} are worked out in (Pattinson, 2003;...
Coinductive Predicates and Final Sequences in a Fibration

Worrell, 2005); the one in (Worrell, 2005) is further extended to locally presentable categories (those are categories suited for speaking of “size”) with additional assumptions in (Adámek, 2003).

Turning back to coinductive predicates, indeed, the fibrational view (Hermida and Jacobs, 1998; Hermida, 1993) identifies coinductive predicates as final coalgebras in a fibration. This leads us to scrutinize final sequences in a fibration. Our main result (Thm. 3.8) is a categorical generalization of the behavioral $\omega$-bound (§1.1)—more precisely we axiomatize categorical “size restrictions” for that bound to hold.

The conditions are formulated in the language of locally presentable categories (see e.g. (Adámek and Rosický, 1994); also Appendix B); and the combination of fibrations and locally presentable categories does not seem to have been studied a lot (an exception is (Makkai and Paré, 1989, §5.3)). We therefore develop a relevant categorical infrastructure (§6). Our results there include a sufficient condition for the total category $\text{Sub}(C)$ of a subobject fibration to be locally (finitely) presentable, and the same for a family fibration $\text{Fam}(\Omega)$. Via these results, in §7 we list some concrete examples of fibrations to which our results in §3 on the behavioral bounds apply. They include:

- $\text{Pred} \downarrow \text{Sets}$ (classical logic);
- $\text{Rel} \downarrow \text{Sets}$ (for bisimulation and bisimilarity);
- $\text{Sub}(C) \downarrow C$ for $C$ that is locally finitely presentable and locally Cartesian closed (a topos is a special case); and
- $\text{Fam}(\Omega) \downarrow \text{Sets}$ for a well-founded algebraic lattice $\Omega$.

1.3. Contributions

To summarize, our contributions are: 1) combination of the mathematical observations in (Hermida, 1993; Hermida and Jacobs, 1998) and (Jacobs, 2012, Chap. 6) for a general formulation of coinductive predicates; 2) categorical behavioral bounds for final sequences that approximate coinductive predicates; and 3) a categorical infrastructure that relates fibrations and locally presentable categories.

Compared to the earlier version (Hasuo et al., 2013) of the current paper, the main differences are as follows. Here we additionally address inductive predicates over coinductive datatypes (see §5). We identify them as coinductive predicates in the fiberwise opposite $\overset{(\text{op})}{P} \downarrow C$ of the original fibration $P \downarrow C$, so that the difference between inductive and coinductive predicates becomes a matter of categorical duality. The examples in §7 are extended accordingly, studying inductive predicates on top of coinductive ones. Besides, we include all the proofs that were omitted in (Hasuo et al., 2013) for space reasons.

1.4. Organization of the Paper

In §2 we identify coinductive predicates as final coalgebras in a fibration, following the ideas of (Hermida, 1993; Hermida and Jacobs, 1998; Jacobs, 2012). The main technical results are in §3, where we axiomatize size restrictions on fibrations and functors for a final sequence to stabilize after $\omega$ steps. These results are reorganized in §4 as a fibration
of invariants. We see in §5, which is added to an earlier version of this paper (Hasuo et al., 2013), that the results in §2–4 apply to inductive predicates too. The next two sections are devoted to examples: firstly in §6 we develop a necessary categorical infrastructure; and then in §7 we discuss concrete examples. In §8 we conclude with some directions of future work. In Appendices we present minimal introductions to the theories of coalgebras, locally presentable categories and fibrations—the three categorical disciplines that our technical developments rely on.

2. Coinductive Predicates as Final Coalgebras

In this section we follow the ideas in (Hermida, 1993; Hermida and Jacobs, 1998; Jacobs, 2012) and characterize coinductive predicates in various settings (for different behavior types, and in various underlying logics) in the language of fibrations. An introduction to fibrations is e.g. in (Jacobs, 1999); see also Appendix C. In this paper for simplicity we focus on poset fibrations. It should however not be hard to move to general fibrations.

Definition 2.1 (Fibration). We refer to poset fibrations (where each fiber is a poset rather than a category) simply as fibrations.

Definition 2.2 (Predicate lifting). Let \( \mathcal{P} \downarrow \mathcal{C} \) be a fibration and \( F \) be an endofunctor on \( \mathcal{C} \). A predicate lifting of \( F \) along \( \mathcal{P} \) is a functor \( \varphi : \mathcal{P} \to \mathcal{P} \) such that \( (\varphi, F) \) is an endomap of fibrations.

\[
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{\varphi} & \mathcal{P} \\
\downarrow & & \downarrow \varphi \\
\mathcal{C} & \xrightarrow{F} & \mathcal{C}
\end{array}
\]  

(5)

This means: that the above diagram commutes; and that \( \varphi \) preserves Cartesian arrows, that is, \( \varphi(f^{*}Q) = (Ff)^{*}(\varphi Q) \). See below.

\[
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{f^{*}} & \mathcal{Q} \\
\downarrow & & \downarrow \varphi \\
\mathcal{C} & \xrightarrow{F} & \mathcal{C}
\end{array}
\]  

(6)

In the prototype example \( \text{Pred}_{\text{Sets}} \downarrow \text{Sets} \), the above definition coincides (see (Jacobs, 2012)) with the one used in coalgebraic modal logic (see e.g. (Cirstea et al., 2011)), the latter being a (monotone) natural transformation \( \varphi(\cdot^{*}) \Rightarrow \varphi(\cdot) \) of fibrations. We think of predicate liftings as (co)recursive definitions of coinductive predicates (see Example 2.4). On top of it, we identify coinductive predicates (and invariants) as coalgebras in a fiber.

Definition 2.3 (Invariant, coinductive predicate). Let \( \varphi \) be a predicate lifting of \( F \) along \( \mathcal{P} \); and \( X \xrightarrow{f} FX \) be a coalgebra in \( \mathcal{C} \). They together induce an endofunctor
Coinductive Predicates and Final Sequences in a Fibration

(a monotone function) on the fiber $\mathbb{P}_X$, namely $\mathbb{P}_X \xrightarrow{\nu} \mathbb{P}_{FX} \xrightarrow{\circ} \mathbb{P}_X$, where $\varphi$ restricts to $\mathbb{P}_X \to \mathbb{P}_{FX}$ because of (5).

1. A $\varphi$-invariant in $c$ is a $(c^* \circ \varphi)$-coalgebra in $\mathbb{P}_X$, that is, an object $P \in \mathbb{P}_X$ such that $P \leq c^*(\varphi P)$ in $\mathbb{P}_X$.

2. The $\varphi$-coinductive predicate in $c$ is the final $(c^* \circ \varphi)$-coalgebra (if it exists). Its carrier shall be denoted by $[\nu \varphi]_c$. It is therefore the largest $\varphi$-invariant in $c$; Lambek’s lemma yields that $[\nu \varphi]_c = (c^* \circ \varphi)([\nu \varphi]_c)$.

Example 2.4 (Rv). The conventional interpretation $[\nu u.\alpha]_c$ (described in §1.1) of Rv-Pred formulas is a special case of Def. 2.3. Indeed, let us work in the fibration $\mathcal{S}_{\text{Sets}}$, and with the endofunctor $F_K = \mathcal{P}(\mathcal{A}P) \times \mathcal{P}(\_)$ on $\mathcal{Sets}$. An $F_K$-coalgebra $X \xrightarrow{\nu} \mathcal{P}(\mathcal{A}P) \times \mathcal{P}X$ is precisely a Kripke model: $c$ combines a valuation $X \to \mathcal{P}(\mathcal{A}P)$ and the map $X \to \mathcal{P}X$ that carries a state to the set of its successors. To each formula $\alpha \in Rv_u$ we associate a predicate lifting $\varphi_\alpha$ of $F_K$. This is done inductively as follows.

\[
\begin{align*}
\varphi_\alpha(U \subseteq X) &= \{ V \in F_K X \mid a \in \pi_1(V) \subseteq F_K X \} \\
\varphi_\pi(U \subseteq X) &= \{ V \in F_K X \mid a \notin \pi_1(V) \subseteq F_K X \} \\
\varphi_\Box u(U \subseteq X) &= \{ V \in F_K X \mid \pi_2(V) \subseteq U \subseteq F_K X \} \\
\varphi_\Diamond u(U \subseteq X) &= \{ V \in F_K X \mid \pi_2(V) \cap U \neq 0 \subseteq F_K X \} \\
\varphi_\alpha \land \alpha'(U \subseteq X) &= \{ \varphi_\alpha U \cap \varphi_\alpha' U \subseteq F_K X \} \\
\varphi_\alpha \lor \alpha'(U \subseteq X) &= \{ \varphi_\alpha U \cup \varphi_\alpha' U \subseteq F_K X \}
\end{align*}
\]

(7)

In the above, $\pi_1$ and $\pi_2$ denote the projections from $F_K X = \mathcal{P}(\mathcal{A}P) \times \mathcal{P}X$. Then it is easily seen by induction that $[\nu \varphi_\alpha]_c$ in Def. 2.3 coincides with the conventional interpretation $[\nu u.\alpha]_c$, described in §1.1.

In fact, the predicate liftings $\varphi_\alpha$ in (7) are the ones commonly used in coalgebraic modal logic (where they are presented as natural transformations). We point out that the same definition of $\varphi_\alpha$—they are written in the internal language of toposes—works for the subobject fibration $\mathbb{P}_X \xrightarrow{\nu} \mathcal{C}$ of any topos $\mathcal{C}$. Therefore the categorical definition of coinductive predicates (Def. 2.3) allows us to interpret the language $Rv$ in constructive underlying logics. Suitable completeness of $\mathcal{C}$ ensures that a final $(c^* \circ \varphi)$-coalgebra in Def. 2.3 exists.

Proposition 2.5. Let $\varphi$ be a predicate lifting of $F$ along $\mathbb{P}_X \xrightarrow{\nu} \mathcal{P}X$ be a coalgebra in $\mathcal{C}$; and $P \in \mathbb{P}_X$. We have $P \leq [\nu \varphi]_c$ if and only if there exists a $\varphi$-invariant $Q$ such that $P \leq Q$. \hfill $\square$

The proposition is trivial but potentially useful. It says that an invariant can be used as a “witness” for a coinductive predicate. This is how bisimilarity is commonly established.
(namely by finding a bisimulation); and it can be used e.g. in (Abramsky and Winschel, 2013, §6) as an alternative to the metric coinduction principle used there.†

Remark 2.6. The coalgebraic modal logic literature exploits the fact that there can be many predicate liftings (in the form of natural transformations) of the same functor $F$. Different predicate liftings correspond to different modalities (such as $\Box$ vs. $\Diamond$ for the same functor $\mathcal{P}$). This view of predicate liftings is also the current paper’s (see Example 2.4).

In contrast, in fibrational studies like (Hermida, 1993; Hermida and Jacobs, 1998; Fumex et al., 2011; Atkey et al., 2012), use of predicate liftings has focused on the validity of the (co)induction proof principle. For such purposes it is necessary to choose a predicate lifting $\varphi$ that is “comprehensive enough,” covering all the possible $F$-behaviors. In fact, it is common in these studies that “the” predicate lifting, denoted by $\text{Pred}(F)$, is assigned to a functor $F$. An exception is (Jacobs, 2010).

3. Final Sequences in a Fibration

Here we present our main technical result (Thm. 3.8). It generalizes known behavioral $\omega$-bounds (like (Hennessy and Milner, 1985, Thm. 2.1); see §1.1); and claims that the chain (2) for a coinductive predicate stabilizes after $\omega$ steps, assuming that the behavior type functor $F$ and the underlying logic $\mathcal{P} \downarrow_{\mathcal{C}} \downarrow_{p}$ are “finitary” in a suitable sense (but no size restriction on $\varphi$).

3.1. Size Restrictions on a Fibration

We axiomatize finitariness conditions in the language of locally presentable categories (see Appendix B for a minimal introduction). Singling out these conditions lies at the heart of our technical contribution.

Definition 3.1 (LFP category). A category $\mathcal{C}$ is locally finitely presentable (LFP) if it is cocomplete and it has a (small) set $\mathcal{F}$ of finitely presentable (FP) objects such that every object is a directed colimit of objects in $\mathcal{F}$.

Definition 3.2 (Finitely determined fibration). A (poset) fibration $\mathcal{P} \downarrow_{\mathcal{C}}$ is finitely determined if it satisfies the following.

1. $\mathcal{C}$ is LFP, with a set $\mathcal{F}$ of FP objects (as in Def. 3.1).
2. $\mathcal{P} \downarrow_{\mathcal{C}}$ has fiberwise limits and colimits.
3. For arbitrary $X \in \mathcal{C}$, let $(X_I)_{I \in I}$ be the canonical diagram for $X$ with respect to $\mathcal{F}$ (i.e. $I = (\mathcal{F} \downarrow_{\mathcal{C}} X)$), with a colimiting cocone $(X_I \cong X)_{I \in I}$. Then for any $P, Q \in \mathcal{P}_X$,

$$P \leq Q \iff \kappa_I^* P \leq \kappa_I^* Q \text{ in } \mathcal{P}_X, \text{ for each } I \in I.$$

† To be precise: only if we take $\text{PE}$ in (Abramsky and Winschel, 2013) as an atomic proposition (and that is essentially what is done in the proofs in (Abramsky and Winschel, 2013, §6)). Our future work on nested $\mu$’s and $\nu$’s will more adequately address the situation.
The intuition behind Cond. 3 is that a predicate $P \in \mathbb{P}_X$ (over arbitrary $X \in \mathbb{C}$) is determined by its restrictions $(\kappa_\ast P)_I$ to FP objects $X_I$. One convenient sufficient condition for Cond. 3 is that the total category $\mathbb{P}$ is itself LFP, with its FP objects residing above the FP objects in $\mathbb{C}$ (Cor. 6.2). We note that Cond. 1 guarantees, since LFP implies completeness, that an $(\omega^{op})$-limit $F^{\omega}1$ of the final $F$-sequence (4) exists. However this does not mean (nor do we need) that $F^{\omega}1$ carries a final $F$-coalgebra; it fails for $F = \mathbb{P}_\omega$, see (Worrell, 2005).

**Definition 3.3 (Well-founded fibration).** A *well-founded fibration* is a finitely determined fibration that further satisfies:

4 If $X \in \mathbb{F}$ (hence FP), the fiber $\mathbb{P}_X$ is such that: the category $\mathbb{P}_X^{op}$ consists solely of FP objects.

Since $\mathbb{P}_X$ is complete, this is equivalent to: there is no $(\omega^{op})$-chain $P_0 > P_1 > \cdots$ in $\mathbb{P}_X$ that is strictly descending.

We note that the following stronger variant of the condition

4' For any $X \in \mathbb{C}$, there is no strictly descending $\omega^{op}$-chain in $\mathbb{P}_X$.

rarely holds (it fails in $\text{Pred} \downarrow \text{Sets}$). The original Cond. 4 holds in many examples (as we will see later in §7) thanks to the restriction that $X$ is FP.

The following trivial fact is written down for the record.

**Lemma 3.4.** A finitely determined fibration $\mathbb{P} \downarrow \mathbb{C}$ is well-founded if $\mathbb{P}_X$ is a finite category for each $X \in \mathbb{F}$. \hfill \Box

### 3.2. Final Sequences in a Fibration

The following result from (Jacobs, 1999, Prop. 9.2.1) is crucial in our development.

**Lemma 3.5.** Let $\mathbb{P} \downarrow \mathbb{C}$ be a fibration, with $\mathbb{C}$ being complete. Then $p$ has fiberwise limits if and only if $\mathbb{P}$ is complete and $p: \mathbb{P} \to \mathbb{C}$ preserves limits. If this is the case, a limit of a small diagram $(P_I)_I$ in $\mathbb{P}$ can be given by

$$\wedge_{I \in \mathbb{I}} (\pi_I^\ast P_I) \quad \text{over } \text{Lim}_{I \in \mathbb{I}} X_I.$$  

Here $X_I := pP_I$; $(\text{Lim}_{I \in \mathbb{I}} X_I \xrightarrow{\tau_{\mathbb{I}}} X_I)_{I \in \mathbb{I}}$ is a limiting cone in $\mathbb{C}$; and $\wedge_{I \in \mathbb{I}}$ denotes the limit in the fiber $\mathbb{P}_{\text{Lim}_{I \in \mathbb{I}} X_I}$. \hfill \Box

Fig. 1 presents two sequences. Here we assume that $\mathbb{P} \downarrow \mathbb{C}$ is well-founded (Def. 3.2) and that $\varphi$ is a predicate lifting of $F$. In the bottom diagram (in $\mathbb{C}$), the object $1 \in \mathbb{C}$ is a final one (it exists since LFP implies completeness); $F1 \xrightarrow{\beta} 1$ is the unique map; $F^{\omega+1}1 := F(F^{\omega}1)$; and $b$ is a unique mediating arrow to the limit $F^{\omega}1$. In the top diagram (in $\mathbb{P}$), the object $\top_1$ is the final object in the fiber $\mathbb{P}_1$; by Lem. 3.5 this is precisely a final object in the total category $\mathbb{P}$. Hence this diagram is nothing but a final sequence for the functor $\varphi$ in $\mathbb{P}$. A limit $\varphi^{\omega+1}_1$ of this final sequence exists, again by Lem. 3.5, and moreover it can be chosen above $F^{\omega}1$. We define $\varphi^{\omega+1}_1 := \varphi(\varphi^{\omega}_1 \top_1)$. 

Lemma 3.6 (Key lemma). Let $\xymatrix{ \mathcal{P} \ar[d]^-p \ar[r] & \mathcal{C} }$ be a well-founded fibration; $F : \mathcal{C} \to \mathcal{C}$ be finitary; and $\varphi$ be a predicate lifting of $F$. Then the final $\varphi$-sequence stabilizes after $\omega$ steps. More precisely: in Fig. 1, we have $\varphi^{\omega+1}1 = b^*(\varphi^\omega 1)$.

Proof. We proceed by steps.

Step a. We observe that, in Fig. 1, the top diagram is carried to the one below by the functor $p : \mathcal{P} \to \mathcal{C}$. This is straightforward: the arrow $\varphi^\omega 1 \to 1$ must be carried to the unique arrow $! : F1 \to 1$; on the mediating arrow $b'$ in $\mathcal{P}$, since $pb'$ is again a mediating arrow in $\mathcal{C}$, it must coincide with $b$.

Step b. Before moving on, we observe that Cond. 3 in Def. 3.2 yields a seemingly stronger statement (Cond. 3' below).

Sublemma 3.7. For a finitely determined fibration $\xymatrix{ \mathcal{P} \ar[d]^-p \ar[r] & \mathcal{C} }$ the following holds.

3' Let $X \in \mathcal{C}; P, Q \in \mathcal{P}_X$; and $(Y_J)_{J \in \mathcal{J}}$ be an arbitrary filtered diagram in $\mathcal{C}$ such that $\operatorname{Colim}_J Y_J = X$, with a colimiting cocone $(Y_J \xrightarrow{\gamma_J} X)_{J \in \mathcal{J}}$. Then $P \leq Q$ if and only if for each $J \in \mathcal{J}$, $\gamma_J^* P \leq \gamma_J^* Q$ in $\mathcal{P}_Y$.

Proof. (Of Sublem. 3.7) The only nontrivial statement is the “if” part of the direction $3 \Rightarrow 3'$. It suffices to show that $\gamma_J^* P \leq \gamma_J^* Q$ (for each $J \in \mathcal{J}$) implies $\kappa_I^* P \leq \kappa_I^* Q$ (for each $I \in \mathcal{I}$), where $\kappa_J$ and $\mathcal{I}$ are as in Cond. 3.

Let $I \in \mathcal{I}$. Since $X_I$ is FP, an arrow $\kappa_I : X_I \to X$ to a filtered colimit $X = \operatorname{Colim}_J Y_J$ factors through some $Y_{I_J} \xrightarrow{\gamma_{I_J}} X$, as in the diagram below.

Now we have $\kappa_I^* P = h_I^* \gamma_{I_J}^* P \leq h_I^* \gamma_{I_J}^* Q = \kappa_I^* Q$, where the inequality is by the assumption that $\gamma_J^* P \leq \gamma_J^* Q$ for each $J \in \mathcal{J}$. This proves Sublem. 3.7.

Step c. By Step a we see that $\varphi^{\omega+1}1 \leq b^*(\varphi^\omega 1)$ by the universality of a Cartesian arrow. In what follows we shall prove its converse:

$$b^*(\varphi^\omega 1) = \varphi^{\omega+1}1 \quad \text{in } \mathcal{P}_{F^{\omega+1}1}.$$  (8)
Let us take a directed diagram \((X_I)_{I \in \mathbb{I}}\) in \(\mathcal{C}\) such that \(X_I \in \mathcal{F}\) (for each \(I \in \mathbb{I}\)) and \(F^\omega 1 = \text{Colim}_{I \in \mathbb{I}} X_I\), with \((X_I \xrightarrow{e_I} F^\omega 1)_{I \in \mathbb{I}}\) being the colimiting cocone. Then we have
\[
F^{\omega + 1} 1 = F(\text{Colim}_{I \in \mathbb{I}} X_I) = \text{Colim}_{I \in \mathbb{I}} F X_I,
\]
by the assumption that \(\mathcal{F}\) is finitary; moreover \((F X_I \xrightarrow{F e_I} F^{\omega + 1} 1)_{I \in \mathbb{I}}\) is a colimiting cocone. The diagram \((X_I)_{I \in \mathbb{I}}\) is directed, and so is the latter diagram \((F X_I)_{I \in \mathbb{I}}\). Thus by Cond. 3' in Sublem. 3.7, showing the following proves (8):
\[
(F \kappa_I)^* (b^* (\varphi^{\omega+1} 1)) \leq (F \kappa_I)^* (\varphi^{\omega+1} 1) \quad \text{for each} \ I \in \mathbb{I}. \quad (9)
\]
**Step d.** To prove (9) we first prove the following fact: for each \(I \in \mathbb{I}\) there exists \(i_I \in \omega\) such that
\[
\kappa^*_I (\varphi^{\omega} 1) = \kappa^*_I (\pi^*_i (\varphi^{\omega} 1)) \quad \text{in} \ P_{X_I}. \quad (10)
\]
That is: the final sequence in \(P\) (Fig. 1), when restricted to \(X_I\) (that is FP), stabilizes within finitely many steps. Indeed, by Lem. 3.5 the limit \(\varphi^{\omega} 1\) is described as an inf in \(P_{F^{\omega-1}}:\)
\[
\varphi^{\omega} 1 = \bigwedge_{i \in \omega} \pi^*_i (\varphi^{\omega} 1). \quad (11)
\]
Therefore we have \(\kappa^*_I (\varphi^{\omega} 1) = \bigwedge_{i \in \omega} \kappa^*_i \pi^*_i (\varphi^{\omega} 1)\) since reindexing \(\kappa^*_i\) preserves fiberwise limits \(\bigwedge\). Now we claim that the sequence \((\kappa^*_i \pi^*_i (\varphi^{\omega} 1))_{i \in \omega}\) in \(P_{X_I}\) is descending: it follows from the fact that \((\pi^*_i (\varphi^{\omega} 1))_{i \in \omega}\) in \(P_{F^{\omega-1}}\) is descending, which in turn is shown from the universality of the Cartesian arrow \(\pi^*_i (\varphi^{\omega} 1)\). See below.

\[
\begin{array}{c}
P \\
\bigwedge_{i \in \omega} \pi^*_i (\varphi^{\omega} 1) \\
\bigwedge_{i \in \omega} \kappa^*_i \pi^*_i (\varphi^{\omega} 1) \\
\kappa^*_I (\bigwedge_{i \in \omega} \pi^*_i (\varphi^{\omega} 1)) = \kappa^*_I (\pi^*_i (\varphi^{\omega} 1)) \quad \text{in} \ P_{X_I}.
\end{array}
\]

Therefore, by \(p\) being a well-founded fibration (Def. 3.3) and \(X_I\) being FP, there exists \(i_I \in \omega\) at which the descending sequence \((\kappa^*_i \pi^*_i (\varphi^{\omega} 1))_{i \in \omega}\) in \(P_{X_I}\) stabilizes, that is,
\[
\kappa^*_I (\bigwedge_{i \in \omega} \pi^*_i (\varphi^{\omega} 1)) = \bigwedge_{i \in \omega} \kappa^*_i \pi^*_i (\varphi^{\omega} 1) = \kappa^*_I (\pi^*_i (\varphi^{\omega} 1)) \quad \text{in} \ P_{X_I}.
\]
Combined with (11), this proves (10).

**Step e.** Finally let us prove (9). For each \(I \in \mathbb{I}\),
\[
(F \kappa_I)^* (b^* (\varphi^{\omega} 1)) = (F \kappa_I)^* (b^* (\bigwedge_{i \in \omega} \pi^*_i (\varphi^{\omega} 1))) \quad \text{by} \ (11)
\]
\[
= \bigwedge_{i \in \omega} (F \kappa_I)^* (b^* (\pi^*_i (\varphi^{\omega} 1))) \quad \text{reindexing preserves} \bigwedge
\]
\[
= \bigwedge_{i \in \omega} (F \kappa_I)^* ((F \pi_{i-1})^* (\varphi^{\omega} 1)) \quad \text{by} \ \pi_i \circ b = F \pi_{i-1} \quad \text{(see Fig. 1)}
\]
\begin{align*}
&= \bigwedge_{i \in \omega} \varphi \left( (\pi_{i-1} \circ \kappa_I)^* (\varphi^{i-1} \top_1) \right) \quad \text{by Def. 2.2} \\
&\leq \varphi \left( (\pi_i \circ \kappa_I)^* (\varphi^i \top_1) \right) \quad \text{letting } i = i_I + 1 \text{ on the LHS} \\
&= \varphi \left( \kappa^*_I \pi^*_I (\varphi^i \top_1) \right) \\
&= (F \kappa_I)^* (\varphi^{i+1} \top_1) \quad \text{by Def. 2.2 and } \varphi^{i+1} \top_1 = \varphi (\varphi^i \top_1). \\
\end{align*}

This proves (9) and concludes the proof of Lem. 3.6.

The object \( \varphi^\omega \top_1 \) is a “prototype” of \( \varphi \)-coinductive predicates in various coalgebras. This is part of the main theorem below.

It is standard that a coalgebra \( X \xrightarrow{c} FX \) in \( \mathcal{C} \) induces a cone over the final \( F \)-sequence, and hence a mediating arrow \( X \xrightarrow{c_\omega} F^\omega 1 \) (see below). Concretely, \( c_i : X \rightarrow F^i 1 \) is defined inductively by: \( X \xrightarrow{c_0} 1 \) is \(!\); and \( c_{i+1} \) is the composite \( X \xrightarrow{c_i} F^i 1 \rightarrow F^{i+1} 1 \). The induced arrow to the limit \( F^\omega 1 \) is denoted by \( c_\omega \).

\begin{equation}
\begin{aligned}
1 &\xleftarrow{\pi_1} F^1 1 \xleftarrow{\cdots} F^i 1 \xleftarrow{\cdots} F^\omega 1 \xleftarrow{c_\omega} X \ \ \\
&\mathop{\Longleftarrow}^{c_i}
\end{aligned}
\end{equation}

Note that \( F^\omega 1 \) does not necessarily carry a final \( F \)-coalgebra (see Remark 3.11).

**Theorem 3.8 (Main result).** Let \( \mathcal{P} \xrightarrow{p} \mathcal{C} \) be a well-founded fibration; \( F : \mathcal{C} \rightarrow \mathcal{C} \) be a finitary functor; \( \varphi \) be a predicate lifting of \( F \) along \( p \); and \( X \xrightarrow{c} FX \) be a coalgebra in \( \mathcal{C} \).

1. **The \( \varphi \)-coinductive predicate \( \nu_\varphi \) in \( c \) (Def. 2.3) exists. It is obtained by the following reindexing of \( \varphi^\omega \top_1 \), where \( c_\omega \) is the mediating map in (12).

\begin{equation}
\nu_\varphi = c_\omega (\varphi^\omega \top_1)
\end{equation}

2. Moreover, the predicate \( \nu_\varphi \) is the limit of the following \( \omega^{op} \)-chain in the fiber \( \mathcal{P}_X \)

\begin{equation}
\top_X \geq (c^* \circ \varphi)(\top_X) \geq (c^* \circ \varphi)^2(\top_X) \geq \cdots,
\end{equation}

that stabilizes after \( \omega \) steps. That is, \( \nu_\varphi = \bigwedge_{i \in \omega} (c^* \circ \varphi)^i(\top_X) \).

**Proof.** To prove the item 1, we proceed by steps.

**Step 1a.** We first show that \( c_\omega (\varphi^\omega \top_1) \) indeed carries a \( (c^* \circ \varphi) \)-coalgebra.

\begin{align*}
&c^* (\varphi (c_\omega (\varphi^\omega \top_1))) \\
&= c^* ((F c_\omega)^* (\varphi (\varphi^\omega \top_1))) \quad \text{by Def. 2.2} \\
&= c^* ((F c_\omega)^* (b^* (\varphi^\omega \top_1))) \quad \text{by Lem. 3.6} \\
&= (b \circ F c_\omega \circ c)^* (\varphi^\omega \top_1) \\
&= c_\omega (\varphi^\omega \top_1). \
\end{align*}
For the last equality we used $b \circ Fc_\omega \circ c = c_\omega$, which is proved by showing that $b \circ Fc_\omega \circ c$ is also a mediating map in (12). Indeed, for each $i \in \omega$,

$$
\pi_i \circ b \circ Fc_\omega \circ c = F\pi_{i-1} \circ Fc_\omega \circ c \quad \text{see Fig. 1}
$$

$$
= Fc_{i-1} \circ c \quad \text{by (12)}
$$

$$
= c_i \quad \text{by def. of } c_i.
$$

**Step 1b.** We show that the coalgebra obtained in Step 1a is final. Let $U \leq c^*(\varphi U)$ be an arbitrary $(c^* \circ \varphi)$-coalgebra (i.e. a $\varphi$-invariant in $c$), where $U \in \mathbb{P}_X$. We aim to establish the following diagram in $P$ and see that it is above the one in (12).

We first show that $U \leq c^*_i (\varphi^i \top_1)$ for each $i \in \omega$. (17)

The proof is by induction. The base case $i = 0$ is obvious since reindexing $c^*_i$ preserves $\top$. For the step case:

$$
U \leq c^*(\varphi(U)) \quad U \text{ carries a } (c^* \circ \varphi)\text{-coalgebra}
$$

$$
\leq c^*(\varphi(c^*_i (\varphi^i \top_1))) \quad \text{by induction hypothesis}
$$

$$
= c^*(FCi)^*(\varphi^{i+1} \top_1) \quad \text{by Def. 2.2}
$$

$$
= (c_{i+1})^*(\varphi^{i+1} \top_1) \quad \text{by def. of } c_{i+1}.
$$

This proves (17) and establishes the arrows $U \rightarrow \varphi^i \top_1$ in (16), for each $i$. Therefore we obtain a mediating map $c'_\omega: U \rightarrow \varphi^\omega \top_1$ to the limit $\varphi^\omega \top_1$, too. The arrow $c'_\omega$ is easily shown to be above $c_\omega$ (much like $b'$ in Fig. 1 is shown to be above $b$); this means $U \leq c^*_\omega (\varphi^\omega \top_1)$. Since $\mathbb{P}_X$ is a poset, this arrow $\leq$ is necessarily a coalgebra morphism from $U$ to $c^*_\omega (\varphi^\omega \top_1)$; moreover it is a unique such. This proves the item 1.

To prove the item 2, we have

$$
[v\varphi] = c^*_\omega (\varphi^\omega \top_1) \quad \text{by the item 1}
$$

$$
= c^*_\omega \left( \bigwedge_{i \in \omega} \pi^*_i (\varphi^i \top_1) \right) \quad \text{by Lem. 3.5}
$$

$$
= \bigwedge_{i \in \omega} c^*_\omega (\pi^*_i (\varphi^i \top_1)) \quad \text{since reindexing preserves } \bigwedge
$$

$$
= \bigwedge_{i \in \omega} c^*_\omega (\varphi^i \top_1) \quad \text{by def. of } c_\omega.
$$

Furthermore, $c^*_\omega (\varphi^i \top_1)$ in the above is seen to be equal to $(c^* \circ \varphi)^i (\top X)$. This is shown by induction on $i \in \omega$. For $i = 0$ the claim amounts to $!^*(\top_1) = \top X$, which holds since
reindexing preserves $\top$. For the step case,

$$
\begin{align*}
    c_{i+1}(\varphi^{i+1} T_1) &= c^*(Fc_i)^*(\varphi^{i+1} T_1) \quad \text{by } c_{i+1} = Fc_i \circ c \\
    &= c^*(\varphi(c_i^*(\varphi^i T_1))) \quad \text{by Def. 2.2} \\
    &= (c^* \circ \varphi)((c^* \circ \varphi)^i(\varphi^i T_1)) \quad \text{by induction hypothesis.}
\end{align*}
$$

Finally let us check that the chain (14) stabilizes after $\omega$ steps.

$$
\begin{align*}
(c^* \circ \varphi)&\left(\bigwedge_{i \in \omega} (c^* \circ \varphi)^i T_X\right) = (c^* \circ \varphi)\left(\bigwedge_{i \in \omega} c_i^*(\varphi^i T_1)\right) \quad \text{by the previous paragraph} \\
    &= (c^* \circ \varphi)(c_\omega^*(\varphi^{\omega} T_1)) \quad \text{by (18)} \\
    &= \bigwedge_{i \in \omega} c_i^*(\varphi^i T_1) \quad \text{by (15)} \\
    &= \bigwedge_{i \in \omega} (c^* \circ \varphi)^i T_X \quad \text{by the previous paragraph.}
\end{align*}
$$

This concludes the proof.

\begin{proof}

Example 3.9 ($\mathbb{R}_\nu$). We shall continue Example 2.4 and derive from Thm. 3.8 the behavioral bound result described in §1.1: the chain (2) stabilizes after $\omega$ steps, for each $\alpha \in \mathbb{R}_\nu u$ and each finitely branching Kripke model $c$.

Indeed, the latter is the same thing as a coalgebra $X \xrightarrow{\nu} F_{\mathbb{R}_K} X$, where $F_{\mathbb{R}_K} = \mathcal{P}(\mathcal{AP}) \times \mathcal{P}_\omega(\_)$, Compared to $F_K$ in Example 2.4 the powerset functor is restricted from $\mathcal{P}$ to $\mathcal{P}_\omega$; this makes $F_{\mathbb{R}_K}$ a finitary functor. Still the same definition of $\varphi_\alpha$ defines a predicate lifting of $F_{\mathbb{R}_K}$. Thm. 3.8.2 can then be applied to the fibration $\text{Pred} \downarrow \text{Sets}$ (easily seen to be well-founded, Example 7.1), the finitary functor $F_{\mathbb{R}_K}$ and the predicate lifting $\varphi_\alpha$ for each $\alpha$. It is not hard to see that the function $[\alpha]_c : \mathcal{P} X \rightarrow \mathcal{P} X$ in §1.1 coincides with $c^* \circ \varphi_\alpha : \text{Pred}_X \rightarrow \text{Pred}_X$ (note that $\text{Pred}_X \cong 2^X \cong \mathcal{P} X$); thus the chain (2) coincides with (14) that stabilizes after $\omega$ steps by Thm. 3.8.

Remark 3.10. The $\omega$-bound of the length of the chain (14) is sharp.

A (counter)example is given in the setting of Example 3.9, by the predicate lifting $\varphi^0 u$ and the coalgebra (i.e. Kripke structure) $c_2$ below. There $b_{i,0}$ has no successors. Indeed, while $[[\nu \varphi^0 u]]_{c_2}$ is $\{a_i \mid i \in \omega\}$, its $i$-th approximant $((c_2)_i^* \circ \varphi^0 u)(\varphi^i T_X)$ in (14) contains $b_{i,0}$ too.

\begin{center}
\begin{tikzpicture}
    \node (a0) at (0,0) {$a_0$};
    \node (b00) at (-2,-1) {$b_{0,0}$};
    \node (b01) at (-1,-1) {$b_{0,1}$};
    \node (b10) at (1,-1) {$b_{1,0}$};
    \node (b11) at (2,-1) {$b_{1,1}$};
    \node (b20) at (3,-1) {$b_{2,0}$};
    \node (b21) at (4,-1) {$b_{2,1}$};
    \node (b22) at (5,-1) {$b_{2,2}$};
    \draw (a0) -- (b00);
    \draw (a0) -- (b01);
    \draw (a0) -- (b10);
    \draw (a0) -- (b11);
    \draw (a0) -- (b20);
    \draw (a0) -- (b21);
    \draw (a0) -- (b22);
    \draw (b00) -- (b01);
    \draw (b10) -- (b11);
    \draw (b20) -- (b21);
    \draw (b21) -- (b22);
\end{tikzpicture}
\end{center}
Coinductive Predicates and Final Sequences in a Fibration

Remark 3.11. It is notable that Thm. 3.8 imposes no size restrictions on $\varphi: \mathcal{P} \to \mathcal{P}$. Being a predicate lifting is enough.

Final $F$-sequences are commonly used for the construction of a final $F$-coalgebra. It is not always the case, however, that the limit $F^\omega 1$ is itself the carrier of a final coalgebra (even for finitary $F$; see (Worrell, 2005, §5)). One obtains a final coalgebra either by: 1) quotienting $F^\omega 1$ by the behavioral equivalence (see e.g. (Pattinson, 2003)); or 2) continuing the final sequence till $\omega + \omega$ steps. The latter construction is worked out in (Worrell, 2005) (in $\mathbf{Sets}$) and in (Adámek, 2003) (in LFP $\mathcal{C}$ with additional assumptions). Its relevance to the current work is yet to be investigated.

Coalgebra morphisms are compatible with coinductive predicates. This fact, like Prop. 2.5, is potentially useful in establishing coinductive predicates.

Proposition 3.12. Let $f: X \to Y$ be a coalgebra morphism from $X \xrightarrow{c} FX$ to $Y \xrightarrow{d} FY$.

In the setting of Lem. 3.6 and Thm. 3.8:

1. If $Q \in \mathcal{P}_Y$ is a $\varphi$-invariant in $d$, so is $f^*Q \in \mathcal{P}_X$ in $c$.
2. We have $[\nu\varphi]_c = f^* ([\nu\varphi]_d)$.

Proof. For the item 1:

\[
\begin{align*}
\text{($f^*Q \leq f^*d^*(\varphi Q))$first invariant} & \leq f^*d^*(\varphi Q) \\
= c^*(Ff)^*(\varphi Q) & \text{by homomorphism} \\
= (c^* \circ \varphi)(f^*Q) & \text{by Def. 2.2.}
\end{align*}
\]

For the item 2, the coalgebras give rise to mediating arrows $X \xrightarrow{c} F^\omega 1$ and $Y \xrightarrow{d} F^\omega 1$, respectively, as in (12). It is easy to see that $c_\omega = d_\omega \circ f$ (using the universality of the limit $F^\omega 1$); using (13) the claim follows.

Remark 3.13. The current paper focuses on finitely presentable objects, finitary functors, etc.—i.e. the $\omega$-presentable setting (see (Adámek and Rosický, 1994, §1.B)). This is for the simplicity of presentation: the results, as usual (as e.g. in (Klin, 2007)), can be easily generalized to the $\lambda$-presentable setting for an arbitrary regular cardinal $\lambda$. In such an extended setting we obtain a behavioral $\lambda$-bound.

4. A Fibration of Invariants

We organize the above observations in a more abstract fibered setting. The technical results are mostly standard; see e.g. (Hermida, 1993; Hermida and Jacobs, 1998) and (Jacobs, 2012, Chap.6).

We write $\mathbf{Coalg}(F)$ for the category of $F$-coalgebras.

Proposition 4.1. Let $\varphi$ be a predicate lifting of $F$ along $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{P}$ along $\mathcal{C}$. Then the fibration $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{P}$ is $\mathbf{Coalg}(\varphi)$ lifted to a fibration $\mathcal{P} \xrightarrow{\mathcal{P}} \mathbf{Coalg}(F)$, with two forgetful functors forming a map of fibrations from the latter to the former.
Proof. It is easy to check each fiber $\text{Coalg}(\varphi)_{X \to FX}$ is a poset. Let $(X \xrightarrow{\varphi} FX) \xrightarrow{f} (Y \xrightarrow{d} FY)$ be an arrow in $\text{Coalg}(F)$, and $P \xrightarrow{s} \varphi P$ be above $Y \xrightarrow{d} FY$. A Cartesian lifting of $f$ are obtained as in the following diagram.

Here we used the universality of the Cartesian lifting $\varphi \mathcal{J}(P)$ (see Def. 2.2).

The two forgetful functors constitute a map of fibrations: the commutativity (5) is obvious, and Cartesian liftings in $\text{Coalg}(\varphi)$ (which we constructed above) are based on the Cartesian liftings in $\mathcal{P}$. \hfill \Box

The next observation explains the current section’s title.

Proposition 4.2. Let $\mathcal{P}$ be the lifted fibration in Prop. 4.1. For each coalgebra $X \xrightarrow{\varphi} FX$, the fiber over $c$ coincides with the poset of $\varphi$-invariants in $c$. That is:

\[
\text{Coalg}(\varphi)_{X \to FX} \to \text{Coalg}(\varphi)_{P \to FX} \\xrightarrow{\varphi \mathcal{J}(P)} \varphi P
\]

Proof. Given a $\varphi$-coalgebra $P \xrightarrow{s} \varphi P$ above $X \xrightarrow{\varphi} FX$, we use the universality of the Cartesian lifting of $c$ to obtain a $(c \circ \varphi)$-coalgebra as in the following diagram.

Conversely, given a $(c \circ \varphi)$-coalgebra $Q \xrightarrow{t} c^*(\varphi Q)$, we obtain a $\varphi$-coalgebra above $X \xrightarrow{\varphi} FX$ as the following composite.

Then it is straightforward to see that the mappings are monotone and inverse to each other. The mappings commute with the forgetful functors since they do not change the carriers. \hfill \Box
Therefore Thm. 3.8.1 and Prop. 3.12.2 state the fibration $\mathcal{P} \downarrow \mathcal{C}$ has fiberwise final objects. (At least part of) this statement itself is shown quite easily using the Knaster-Tarski theorem (each fiber is a complete lattice). Our contribution is their concrete construction as $\omega^{op}$-limits (Thm. 3.8.2).

The following lemma is essentially a special case of Lem. 3.5, but see also (Jacobs, 1999, Prop. 9.2.1 and Exercise 9.2.4).

**Lemma 4.3.** Let $\mathcal{P} \downarrow \mathcal{C}$ be a fibration; and assume that $\mathcal{C}$ has a final object. Then $\mathcal{P} \downarrow \mathcal{C}$ has a fiberwise final object if and only if $\mathcal{P}$ has a final object that is above the final object of $\mathcal{C}$.

By applying the lemma to $\mathcal{P} \downarrow \mathcal{C}$, we obtain a basic relationship between coinductive predicates and final coalgebras.

**Corollary 4.4.** Let $\varphi$ be a predicate lifting of $F$ along $\mathcal{P} \downarrow \mathcal{C}$; and assume that a final $F$-coalgebra exists. The following are equivalent.

1. The coinductive predicate $[\nu \varphi]_c$ exists for each coalgebra $c: X \to FX$. Moreover they are preserved by reindexing (along coalgebra morphisms).
2. There exists a final $\varphi$-coalgebra that is above the final $F$-coalgebra.

As noted in Remark 3.11, however, our concrete construction of coinductive predicates does not rely on a final $F$-coalgebra.

## 5. Inductive predicates over coinductive datatypes

The central topic of the current paper is coinductive predicates over coinductive datatypes, the latter identified as coalgebras in the base category $\mathcal{C}$ of a fibration $\mathcal{P} \downarrow \mathcal{C}$. Some variations are possible, namely: inductive/coinductive predicates over inductive/coinductive datatypes. For example, (Hermida and Jacobs, 1998) focus on: inductive predicates over inductive datatypes (the latter identified as algebras); and coinductive predicates over coinductive datatypes (as we have done in the previous sections).

It turns out that, among these four variations, inductive predicates over coinductive datatypes allow a straightforward adaptation of our current categorical framework by taking the fiberwise opposite $\mathcal{P}^{(op)} \downarrow \mathcal{C}$ of the fibration $\mathcal{P} \downarrow \mathcal{C}$ we are interested in. We present these results in the current section. The study of the other two variations—inductive predicates over inductive datatypes, and coinductive predicates over inductive datatypes—is left as future work. In fact we have preliminary observations that under certain assumptions these two variations coincide. Their details will be presented in another venue.

The following is the definition of an inductive predicate (on a coinductive datatype). It is not hard to see that the definition generalizes e.g. the semantics of the $\mu$ operator of the modal $\mu$-calculus in a Kripke model. Later in Lem. 5.4 we will identify it as a coinductive predicate in the fiberwise opposite.
Definition 5.1 (Inductive predicate). Let \( \varphi \) be a predicate lifting along a fibration \( P \downarrow p \); and \( X \rightarrow F \) be a coalgebra in \( C \). The \( \varphi \)-inductive predicate in \( c \) is the initial \( (c^* \circ \varphi) \)-algebra (if it exists). We denote its carrier by \( J_{\mu \varphi} \). Hence, it is the smallest predicate \( P \in \mathbb{P}_X \) such that \( P \geq c^* (\varphi P) \) in \( \mathbb{P}_X \).

In what follows we utilize the notion of fiberwise opposite \( P^{(\text{op})} \downarrow p^{(\text{op})} \) of a fibration \( P \downarrow p \) (((Bénabou, 1975); see also (Jacobs, 1999, Def. 1.10.11)). Intuitively, the fiberwise opposite \( p^{(\text{op})} \) is obtained by opposing the order in each fiber \( \mathbb{P}_X \) but leaving the base category \( C \), as well as the reindexing structure, as in the original fibration \( p \). The precise definition is best stated via indexed categories and the Grothendieck construction. It is left to the appendix (Lem. C.10).

Some remarks are in order. Firstly, the total category \( \mathbb{P}^{(\text{op})} \) of the fiberwise opposite \( P^{(\text{op})} \downarrow p^{(\text{op})} \) is in general different from the opposite category \( \mathbb{P}^{\text{op}} \) (in the usual sense) of \( \mathbb{P} \). The same applies to the functor \( p^{(\text{op})} \), that is different from the opposite functor \( p^{\text{op}} \).

We emphasize that in the fiberwise opposite \( P^{(\text{op})} \downarrow p^{(\text{op})} \), the base category \( C \) stays the same.

We also note that \( P^{(\text{op})} \downarrow p^{(\text{op})} \) is a fibration, unlike the opposite functor \( P^{\text{op}} \downarrow p^{\text{op}} \) of \( p \) that is canonically an opfibration.

Notation 5.2. For distinction, we denote reindexing functors in fibrations \( P \downarrow p \) and \( P^{(\text{op})} \downarrow p^{(\text{op})} \) by \( f^* \) and \( f^\# \), respectively. They are in fact the same monotone functions between fibers as posets.

The following result, although straightforward, is essential for the subsequent technical development.

Lemma 5.3. Let \( P \downarrow p \) be a fibration and \( F \) be an endofunctor on \( C \). For a predicate lifting \( \varphi : P \rightarrow P \) of \( F \) along \( p \), there exists a canonical predicate lifting \( \varphi^{(\text{op})} : P^{(\text{op})} \rightarrow P^{(\text{op})} \), which we call the fiberwise opposite of \( \varphi \), of \( F \) along the fibration \( P^{(\text{op})} \downarrow p^{(\text{op})} \).

Proof. We give an explicit construction here, although the statement is almost trivial when stated in terms of indexed categories.

On objects, we define \( \varphi^{(\text{op})} P = \varphi P \). For the action on arrows, we first note that an arrow \( P \rightarrow Q \) in \( \mathbb{P}^{(\text{op})} \) above \( f : X \rightarrow Y \) exists if and only if \( P \leq f^# Q \) in \( (\mathbb{P}^{(\text{op})})_X = (P_X)^{\text{op}} \). Exploiting this fact, \( \varphi^{(\text{op})} \)'s action on the arrow \( P \rightarrow Q \) is defined to be the unique arrow \( \varphi^{(\text{op})} P \rightarrow \varphi^{(\text{op})} Q \) above \( Ff : FX \rightarrow FY \). The last (unique) arrow exists, indeed: we have \( \varphi^{(\text{op})} P \leq (Ff)^# \varphi^{(\text{op})} Q \) in \( (\mathbb{P}^{(\text{op})})_F \) by \( \varphi P \geq \varphi f^# Q = (Ff)^* \varphi Q \) in \( \mathbb{P}_F \). Here the last equality is because \( \varphi \) is a predicate lifting. \( \square \)

Lemma 5.4. Let \( P \) be a predicate over \( X \in C \).

1 The object \( P \in \mathbb{P}_X \) carries a \( (c^* \circ \varphi) \)-algebra if and only if \( P \in (\mathbb{P}^X)^{\text{op}} = (\mathbb{P}^{(\text{op})})_X \) is a \( \varphi^{(\text{op})} \)-invariant in \( c \).
The $\varphi$-inductive predicate in $c$ is the $\varphi^{\text{op}}$-coinductive predicate in $c$. That is, $[\mu \varphi]_c = [\nu(\varphi^{\text{op}})]_c$ as objects in $\mathbb{P}_X$.

**Proof.** The category of $(c^* \circ \varphi)$-algebras in $\mathbb{P}_X$ is dually equivalent to the category of $(c^# \circ \varphi^{\text{op}})$-coalgebras in $(\mathbb{P}^{\text{op}})_X$, since the following diagram (in Posets) commutes.

\[
\begin{array}{ccc}
(P^{\text{op}})_X & \xrightarrow{\varphi^{\text{op}}} & (P^{\text{op}})_F \xrightarrow{c^#} (P^{\text{op}})_X \\
\parallel & & \parallel \\
(P_X)^{\text{op}} & \xrightarrow{\varphi_X^{\text{op}}} & (P_{F,X})^{\text{op}} \xrightarrow{(c^*)^{\text{op}}} (P_{X})^{\text{op}}
\end{array}
\]

Thanks to the previous characterization—inductive predicates in $\mathbb{P}^{\downarrow}C$ as coinductive ones in $\mathbb{P}^{\text{op}}(\downarrow)C$—we can apply all the results that we have obtained so far to inductive predicates. Notice again that the base category $C$ has remained the same. The characterization in Lem. 5.4 can be seen as a generalization of the duality $\mu u \varphi(u) = \neg \nu u \neg \varphi(\neg u)$ between least and greatest fixed points in classical logics—the latter is a special case where fibers are self-dual, i.e. $\mathbb{P}^{\text{op}} \cong (\mathbb{P}^{\text{op}})^{\text{op}}$.

Via the last characterization, our main result (Thm. 3.8) can also be used to show the stabilization of the $\omega$-chain when calculating inductive predicates (see Cor. 5.8). The inductive predicate on $F^{\omega}1$ is not a limit nor a colimit in $\mathbb{P}$, but it is a limit in $\mathbb{P}^{\text{op}}$ (see Def. 5.7).

**Definition 5.5 (Co-well-founded fibration).** A co-well-founded fibration is a finitely determined fibration that further satisfies:

1. If $X \in \mathbb{F}$ (hence FP), the fiber $\mathbb{P}_X$ is such that: the category $\mathbb{P}_X$ consists solely of FP objects.

Since $\mathbb{P}_X$ is cocomplete, this is equivalent to: there is no $(\omega)$-chain $P_0 < P_1 < \cdots$ in $\mathbb{P}_X$ that is strictly ascending.

**Lemma 5.6.** For a finitely determined fibration $\mathbb{P}^{\downarrow}C$, its fiberwise opposite $\mathbb{P}^{\text{op}}(\downarrow)C$ is also finitely determined. Moreover, $p^{\text{op}}$ is well-founded if and only if the fibration $p$ is co-well-founded.

**Proof.** It is trivial that the fibration $p^{\text{op}}$ satisfies the condition 1 (of Def. 3.2) if and only if $p$ satisfies it. For the condition 2, $p^{\text{op}}$ has fiberwise limits and colimits, because $p$ has fiberwise colimits and limits, respectively. The condition 3 for $p^{\text{op}}$ is obviously equivalent to the one for $p$ since reindexing functors $\kappa^*_C, \kappa_F^*$ are the same as functions. By $(\mathbb{P}^{\text{op}})_X = (\mathbb{P}_X)^{\text{op}}$, $p^{\text{op}}$ satisfies the condition 4 if and only if $p$ satisfies $\mathbb{P}$.

**Definition 5.7.** Let $\perp_1$ be the least element of the fiber $\mathbb{P}_1$ (hence the greatest in $(\mathbb{P}^{\text{op}})_1$). We denote by $\varphi^{\omega} \perp_1 \in \mathbb{P}_{F^{\omega}1}$ the limit of the following diagram in $\mathbb{P}^{\text{op}}$. It is easily seen to reside above the final $F$-sequence in $\mathbb{C}$.

$$\begin{array}{ccccccc}
\perp_1 & \leftarrow & \varphi \perp_1 & \leftarrow & \cdots & \leftarrow & \varphi^{\omega} \perp_1 & \leftarrow & \cdots \\
\text{in } \mathbb{P}^{\text{op}}
\end{array}$$
Note here that $\perp_1$ is the final object in $\mathbb{P}^{\text{op}}$, and the object $\varphi_1$ is the functor $\varphi^{\text{op}}$ applied to $\perp_1$. Therefore the above diagram is the final $\varphi^{\text{op}}$-sequence in $\mathbb{P}^{\text{op}}$.

Using Lem. 3.5, it is not hard to see that $\varphi^\omega \perp_1 = \bigvee_{i \in \omega} \pi_1^i(\varphi^i \perp_1)$ in the fibration $\mathbb{P}^{\text{op}}$, where $(\pi_i : F^i \perp_1 \to F^1)_{i \in \omega}$ is the limiting cone for the final $F$-sequence in $C$.

The following is our main result adapted to inductive predicates. In particular it states that an inductive predicate is computed as a supremum of an $\omega$-chain.

**Corollary 5.8.** Let $\mathbb{P}^{\text{op}} : C \to C$ be a co-well-founded fibration; $F : C \to C$ be a finitary functor; $\varphi$ be a predicate lifting of $F$ along $\mathbb{P}$; and $X \rightarrowtail FX$ be a coalgebra in $C$.

1. The $\varphi$-inductive predicate $[\mu \varphi]_c$ in $c$ exists. It is obtained by the following reindexing of $\varphi^\omega \perp_1$, where $c_\omega$ is the mediating map in (12).

$[\mu \varphi]_c = c_\omega^* (\varphi^\omega \perp_1)$

2. Moreover, the predicate $[\mu \varphi]_c$ is the colimit of the following $\omega$-chain in the fiber $\mathbb{P}_X$

$\perp_X \leq (c^* \circ \varphi)(\perp_X) \leq (c^* \circ \varphi)^2(\perp_X) \leq \cdots$, 

that stabilizes after $\omega$ steps. That is, $[\mu \varphi]_c = \bigvee_{i \in \omega} (c^* \circ \varphi)^i(\perp_X)$.

**Proof.** By Lem. 5.4, Lem. 5.6, and Thm. 3.8.

**Corollary 5.9.** Let $\varphi$ be a predicate lifting of $F$ along $\mathbb{P}^{\text{op}} : C \to C$; and $(\mathbb{Coalg}(\varphi^{(\text{op})})^{(\text{op})})^{(\text{op})}$ be the fiberwise opposite of the lift of the fibration $\mathbb{P}^{\text{op}}$ (see Prop. 4.1 and Lem. 5.3). For each coalgebra $X \rightarrowtail FX$, the following diagram commutes.

**Proof.** Apply Prop. 4.2 for the predicate lifting $\varphi^{(\text{op})}$ along $\mathbb{P}^{\text{op}}$, we obtain

$\text{Coalg}(\varphi^{(\text{op})})_X \rightarrowtail FX \rightarrowtail \text{Alg}(c^* \circ \varphi)$

whose opposite categories are the ones in the diagram we want to prove.

Coalgebra morphisms are compatible with inductive predicates just as in Prop. 3.12. Therefore the inductive predicates $[\mu \varphi]_c$ form a fiberwise initial object $\bot = [\mu \varphi]$ of the fibration $\mathbb{P}^{(\text{op})}$. 
6. Examples at Large

Here are several results that ensure a fibration to be finitely determined or well-founded, and hence enable us to apply Thm. 3.8. Some of them are well-known; others—especially those which relate fibrations and locally (finitely) presentable categories, including Lem. 6.4 and Cor. 6.10—seem to be new.

The following results provide sufficient conditions for a fibration to be finitely determined (Def. 3.2). Recall that a full subcategory \( P \) of \( \mathcal{P} \) is said to be dense if each object \( P \in \mathcal{P} \) is a colimit of a diagram in \( \mathcal{F} \).

**Lemma 6.1.** Let \( \mathcal{P} \) be a fibration with fiberwise limits and colimits and coproducts \( \bigsqcup \) between fibers. Assume further that \( \mathcal{C} \) is LFP with a set \( \mathcal{F}_\mathcal{C} \) of FP objects (as in Def. 3.1). If the total category \( \mathcal{P} \) has a dense subcategory \( \mathcal{F}_\mathcal{P} \) such that every \( R \in \mathcal{F}_\mathcal{P} \) is above \( \mathcal{F}_\mathcal{C} \) (i.e. \( pR \in \mathcal{F}_\mathcal{C} \)), then \( p \) is finitely determined.

**Proof.** The only nontrivial part is the \( \Leftarrow \) direction of Cond. 3. For that it suffices to show that arbitrary \( P \in \mathcal{P} \) is a colimit of the diagram \( (\kappa^*_I P)_{I \in I} \). Here \( I \) and \( \kappa_I \) are as in Cond. 3.

By Lem. C.9 the colimit \( \operatorname{Colim}_{I \in I} \kappa^*_I P \) is described as \( \bigsqcup_{I \in I} \kappa^*_I \kappa^*_I P \) using a sup \( \bigsqcup \) in \( \mathcal{P}_X \), since \( (X_I \xrightarrow{\kappa_I} X)_{I \in I} \) is colimiting. We have \( \bigsqcup_{I \in I} \kappa^*_I P \leq P \) as a counit of an adjunction; therefore \( \operatorname{Colim}_{I \in I} \kappa^*_I P \leq P \).

Thus it suffices to show that \( P \leq \operatorname{Colim}_{I \in I} \kappa^*_I P \) in \( \mathcal{P}_X \). Let \( (P_J)_{J \in J} \) be a diagram in \( \mathcal{P} \) such that \( P_J \in \mathcal{F}_\mathcal{P} \) and there is a colimiting cocone \( (P_J \xrightarrow{g_J} P)_{J \in J} \). Such a diagram exists since \( \mathcal{F}_\mathcal{P} \) is dense.

By the assumption, for each \( J \) the object \( P_J \in \mathcal{F}_\mathcal{P} \) lies above an object in \( \mathcal{F}_\mathcal{C} \). Therefore the arrow \( pg_J : pP_J \to pP = X \) is an object of \( (\mathcal{F}_\mathcal{C} \downarrow X) \); since \( I \) and \( (\mathcal{F}_\mathcal{C} \downarrow X) \), we can choose \( I_J \in I \) such that \( \kappa_{I,J} = pg_J \).

Now an arrow \( P_J \xrightarrow{g_J} P \) in \( \mathcal{P} \) induces \( P_J \leq (pg_J)^* P = \kappa^*_J P \) (19) by the universality of Cartesian arrows. We proceed as follows.

\[
P = \operatorname{Colim}_{J \in J} P_J \overset{(*)}{=} \bigvee_{J \in J} \operatorname{Colim}_{I \in I} \kappa^*_I P_J \overset{(*)}{=} \bigvee_{J \in J} \bigsqcup_{I \in I} \kappa^*_I P_J \leq \bigsqcup_{I \in I} \bigsqcup_{I \in I} \kappa^*_I P = \operatorname{Colim}\kappa^*_I P .
\]

For \( (*) \) we used Lem. C.9; \( (\dagger) \) holds since \( I_J \) is chosen so that \( \kappa_{I,J} = pg_J \) and (19) hold. This concludes the proof.

**Corollary 6.2.** Let \( \mathcal{P} \) be a fibration with fiberwise limits and colimits and coproducts \( \bigsqcup \) between fibers, where \( \mathcal{C} \) is LFP with a set \( \mathcal{F}_\mathcal{C} \) of FP objects (in Def. 3.1). If the total category \( \mathcal{P} \) is also LFP, with a set \( \mathcal{F}_\mathcal{P} \) of FP objects (as in Def. 3.1) chosen so that every \( R \in \mathcal{F}_P \) is above \( \mathcal{F}_\mathcal{C} \), then \( p \) is finitely determined.

6.1. Subobject Fibrations

We first note a well-known result.
Lemma 6.3 (Jacobs, 1999, Prop. 5.4.7). An (elementary) topos is a locally Cartesian closed category (LCCC).

The following is one of the results that are less trivial.

Lemma 6.4. Let $\mathcal{C}$ be an LFP category with $F$ being a set of FP objects (as in Def. 3.1). Assume that $\mathcal{C}$ is at the same time an LCCC. Then the total category $\text{Sub}(\mathcal{C})$ of the subobject fibration is LFP: the set $\mathbb{F}_{\text{Sub}(\mathcal{C})} := \{(P \rightarrow X) \mid P, X \in F\}$ consists of FP objects in $\text{Sub}(\mathcal{C})$; and every object $(Q \rightarrow Y) \in \text{Sub}(\mathcal{C})$ is a colimit of a directed diagram in $\mathbb{F}_{\text{Sub}(\mathcal{C})}$.

Proof. The proof is by steps.

Step a. First we show that $\text{Sub}(\mathcal{C})$ is complete and cocomplete. We rely on Lem. C.9.

We start with fiberwise limits in $\text{Sub}(\mathcal{C}) \downarrow \mathcal{C}$; the proof is like in (Jacobs, 1999, Example 1.8.3(iii)). By Lem. B.4 an LFP category $\mathcal{C}$ is complete. This equips each fiber $\text{Sub}(X)$ with arbitrary inf's $\bigwedge$ computed as wide pullbacks. A reindexing functor (by pullbacks) preserves these inf's since limits commute. Therefore by Lem. C.9 the total category $\text{Sub}(\mathcal{C})$ is complete.

By the assumption that $\mathcal{C}$ is an LCCC, $\text{Sub}(\mathcal{C}) \downarrow \mathcal{C}$ has products $\prod f \vdash f^*$ between fibers (Jacobs, 1999, Cor. 1.9.9).

Next we show that $\text{Sub}(\mathcal{C}) \downarrow \mathcal{C}$ has fiberwise colimits. Each fiber (which is a poset) has arbitrary inf's; hence it is a complete lattice and arbitrary sup's also exist. These sup's (i.e. colimits in a fiber) are preserved by reindexing $f^*$ since the latter is a left adjoint $f^* \dashv \prod f$.

We further show that $\text{Sub}(\mathcal{C}) \downarrow \mathcal{C}$ has coproducts $\bigsqcup$ between fibers. An abstract proof can be given by Freyd's adjoint functor theorem (note that each fiber $\text{Sub}(X)$ is a complete lattice, and that reindexing $f^*$ preserves inf's). Instead we explicitly introduce $\bigsqcup$ exploiting a factorization structure of LFP $\mathcal{C}$ (Lem. B.4). Namely, given $(P \xrightarrow{m} X) \in \text{Sub}(X)$ and $f : X \rightarrow Y$, the coproduct $\bigsqcup_f P$ is defined by the (StrongEpi, Mono)-factorization of $f \circ m$, as below.

![Diagram](20)

The fact that $\bigsqcup_f P \leq Q$ if and only if $P \leq f^*Q$ is easily proved using the diagonalization property of the factorization structure. This establishes $\bigsqcup_f$ as a left adjoint to reindexing $f^*$. These coproducts $\bigsqcup$ satisfy the Beck-Chevalley condition since the products $\prod$ do (Jacobs, 1999, Lem. 1.9.7). Using Lem. C.9 we conclude that $\text{Sub}(\mathcal{C})$ is cocomplete.

Step b. First we prove that, if $P$ and $X$ are both FP in $\mathcal{C}$, then $(P \xrightarrow{m} X)$ is FP in $\text{Sub}(\mathcal{C})$. Let $(Q_t \xrightarrow{n_t} Y_t)_{t \in I}$ be a filtered diagram in $\text{Sub}(\mathcal{C})$; $(Q \xrightarrow{n} Y)$ its colimit; and $g : (P \xrightarrow{m} X) \rightarrow (Q \xrightarrow{n} Y)$ an arrow in $\text{Sub}(\mathcal{C})$. By Lem. C.9 the colimit $(Q \xrightarrow{n} Y)$ can be
Coinductive Predicates and Final Sequences in a Fibration

explicitly described as

\[ Y = \text{Colim}_{I \in I} Y_I , \quad Q = \bigvee_{I \in I} \prod_{\kappa_I} Q_I , \quad (21) \]

where \((Y_I \xrightarrow{\kappa_I} Y)_{I \in I}\) is a colimiting cocone.

**Sublemma 6.5.** The object \(Q \in C\) is a colimit \(\text{Colim}_{I \in I} Q_I\) computed in \(C\).

**Proof.** (Of the sublemma) Both \((Q_I)_{I \in I}\) and \((Y_I)_{I \in I}\) are \(I\)-shaped diagrams in \(C\) with a monotransformation \((Q_I \xrightarrow{n_I} Y_I)_I\). Therefore by (Adámek and Rosický, 1994, Cor. 1.60), the induced arrow \(\text{Colim}_I Q_I \rightarrow \text{Colim}_I Y_I\) is monic, establishing \(\text{Colim}_I Q_I \in \text{Sub}(Y)\).

It suffices to find arrows \(a, b\) in the diagram below.

\[
\begin{array}{ccc}
\text{Colim}_I Q_I & \xrightarrow{a} & Q \\
\downarrow b & & \downarrow n \\
Y & \xrightarrow{\kappa_I} & \text{Colim}_I Y_I \\
\end{array}
\]

(22)

The arrow \(a\) is obtained in the following way. Since \(\kappa_I\) is an arrow \((Q_I \xrightarrow{\kappa_I} Y_I)_I\) in \(\text{Sub}(C)\), by (34) we have an arrow \(Q_I \rightarrow Q\) in \(C\), for each \(I \in \mathbb{I}\). These arrows induce \(a\) as a mediating arrow.

To obtain \(b\) in (22), since \(Q = \bigvee_{I \in I} \prod_{\kappa_I} Q_I\) (see (21)), it suffices to find \(b_I\) below for each \(I \in \mathbb{I}\).

\[
\begin{array}{ccc}
\prod_{\kappa_I} Q_I & \xrightarrow{b_I} & \text{Colim}_I Q_I \\
\downarrow Y & \xleftarrow{\kappa_I} & \downarrow \text{Colim}_I Y_I \\
\end{array}
\]

This is obtained as the following diagonal fill-in. Recall that \(Y = \text{Colim}_I Y_I\).

\[
\begin{array}{ccc}
Q_I & \xrightarrow{n_I} & \prod_{\kappa_I} Q_I \\
\downarrow & \xleftarrow{b_I} & \downarrow \text{Colim}_I Q_I \\
\text{Colim}_I Q_I & \xrightarrow{[n_I, \kappa_I]} & \text{Colim}_I Y_I \\
\end{array}
\]

This proves Sublem. 6.5. \(\square\)

We are back in Step 6. Since \(g: (P \xrightarrow{m} X) \rightarrow (Q \xrightarrow{n} Y)\) is an arrow in \(\text{Sub}(C)\), we also have an arrow \(g': P \rightarrow Q\), such that \(n \circ g' = g \circ m\), by (34). Now \(Q\) and \(Y\) are filtered colimits of \((Q_I)_{I \in I}\) and \((Y_I)_{I \in I}\), respectively (the former is by Sublem. 6.5). Since \(P\) and \(X\) are FP, \(I_0 \in \mathbb{I}\) can be chosen such that \(g\) factors through \(Y_{I_0} \rightarrow Y\) and \(g'\) factors through \(Q_{I_0} \rightarrow Q\). That is,

\[
\begin{array}{ccc}
P & \xrightarrow{h'} & Q_{I_0} \\
\downarrow m & & \downarrow n \\
X & \xrightarrow{h} & Y_{I_0} \\
\downarrow g & & \downarrow \kappa_{I_0} \\
& & Y \\
\end{array}
\]
It is not (yet) necessarily the case that the square on the left commutes, i.e. \( n_{I_0} \circ h' = h \circ m \). The two arrows give factorizations of the arrow \( n \circ g' = g \circ m \): \( P \rightarrow Y \) via \( Y_{I_0} \overset{\kappa_{I_0}}{\rightarrow} Y \); since \( P \) is FP, there exists \( I_1 \in I \) with \( i: I_0 \rightarrow I_1 \) such that

\[
(Y_i) \circ n_{I_0} \circ h' = (Y_i) \circ h \circ m
\]

(essential uniqueness of factorization, (Adámek and Rosický, 1994, Def. 1.1)). It is clear that, for such \( I_1 \), the arrow \( g \) in \( \text{Sub}(\mathbb{C}) \) factors through \( \langle Q_{I_1} \rightarrow Y_{I_1} \rangle \overset{\kappa_{I_1}}{\rightarrow} \langle Q \rightarrow Y \rangle \). This concludes Step b that \( (P \rightarrow X) \) is FP in \( \text{Sub}(\mathbb{C}) \).

**Step c.** Recall that

\[
\mathbb{F}_{\text{Sub}(\mathbb{C})} := \{ \langle P \rightarrow X \rangle \mid P, X \in \mathbb{F} \}.
\]  

(23)

The set \( \mathbb{F}_{\text{Sub}(\mathbb{C})} \) in (23) is small, since \( \mathbb{F} \) is small and \( \mathbb{C} \) is well-powered (Adámek and Rosický, 1994, Rem. 1.56).

**Step d.** In the remainder of the proof we show that every object \( \langle Q \overset{n}{\rightarrow} Y \rangle \in \text{Sub}(\mathbb{C}) \) is a colimit of the canonical diagram with respect to \( \mathbb{F}_{\text{Sub}(\mathbb{C})} \) from (23). Let \( \langle Q_J \overset{n_J}{\rightarrow} Y_J \rangle_{J \in \mathbb{J}} \) be the canonical diagram (i.e. \( \mathbb{J} = (\mathbb{F}_{\text{Sub}(\mathbb{C})} \downarrow n) \)), with the canonical cocone

\[
(\langle Q_J \overset{n_J}{\rightarrow} Y_J \rangle \overset{f_J}{\rightarrow} \langle Q \overset{n}{\rightarrow} Y \rangle)_{J \in \mathbb{J}}.
\]  

(24)

Let us denote the canonical diagram for \( Y \in \mathbb{C} \) with respect to \( \mathbb{F} \) by \( \langle Y’_I \overset{k_I}{\rightarrow} Y \rangle_{I \in \mathbb{I}} \) (i.e. \( \mathbb{I} = (\mathbb{F} \downarrow Y) \)), with a canonical cocone \( \langle Y’_I \overset{k_I}{\rightarrow} Y \rangle_{I \in \mathbb{I}} \) since \( \mathbb{C} \) is LFP. In this Step d we show \( \text{Colim}_{J \in \mathbb{J}} Y_J \cong \text{Colim}_{I \in \mathbb{I}} Y’_I = Y \). A cocone \( \langle Y_J \overset{n_J}{\rightarrow} \text{Colim}_{I \in \mathbb{I}} Y’_I \rangle_{J \in \mathbb{J}} \) can be defined by finding (unique) \( I_J \in I \) such that \( Y’_{I_J} \overset{f_{I_J}}{\rightarrow} Y \) is equal to \( Y_J \overset{f_J}{\rightarrow} Y \). In order to see that this cocone is colimiting, let \( \langle Y_J \overset{g_J}{\rightarrow} Z \rangle_{J \in \mathbb{J}} \) be another cocone (recall that \( \mathbb{J} = (\mathbb{F}_{\text{Sub}(\mathbb{C})} \downarrow n) \)).

**Sublemma 6.6.** If the indices \( J, J’ \in \mathbb{J} \) satisfy \( Y_J = Y_{J’} \) and \( f_J = f_{J’} \) (cf. (24)), then \( g_J = g_{J’} \).

**Proof.** (Of the sublemma) Let \( 0 \) be an initial object in \( \mathbb{C} \). We first prove that \( 0 \) is a subobject of any object of \( \mathbb{C} \). Indeed, since \( \mathbb{C} \) is an LCCC (hence a CCC), its initial object \( 0 \) is strict, meaning that if \( U \rightarrow 0 \) exists then \( U \) is also initial (a standard result; see e.g. (Johnstone, 2002, Lem. 1.5.12)). This makes any arrow \( 0 \rightarrow V \) in \( \mathbb{C} \) a mono.

Therefore we have an object \( 0 \rightarrow Y_J \) in \( \text{Sub}(\mathbb{C}) \) induced by initiality. In view of (34), an arrow \( f_J \in \mathbb{C} \) induces an arrow

\[
(0 \rightarrow Y_J) \overset{f_J}{\rightarrow} (Q \overset{n}{\rightarrow} Y) \quad \text{in} \quad \text{Sub}(\mathbb{C});
\]

therefore there exists an index \( J” \in \mathbb{J} = (\mathbb{F}_{\text{Sub}(\mathbb{C})} \downarrow n) \) such that \( 0 \rightarrow Y_{J”} = (Q_{J”} \overset{n_{J”}}{\rightarrow} Y_{J”}) \), and \( f_{J”} = f_J = f_{J’} \).

This gives us the following diagram in \( \mathbb{J} = (\mathbb{F}_{\text{Sub}(\mathbb{C})} \downarrow n) \).

\[
(Q_J \overset{n_J}{\rightarrow} Y_J) \overset{\text{id}_{Y_J}}{\rightarrow} (Q_{J”} \overset{n_{J”}}{\rightarrow} Y_{J”}) \overset{\text{id}_{Y_{J”}}}{\rightarrow} (Q_{J’} \overset{n_{J’}}{\rightarrow} Y_{J’})
\]
Therefore, since \((Y_J \xrightarrow{g_J} Z)_{J \in J}\) is a cocone, the following diagram in \(C\) must commute.

\[
\begin{array}{c}
Y_J \\
\downarrow g_J \\
Z
\end{array}
\begin{array}{c}
\downarrow g_J' \\
\downarrow g_J''
\end{array}
\begin{array}{c}
\xrightarrow{id} Y_J' \\
\xrightarrow{id} Y_J''
\end{array}
\]

This proves \(g_J = g_{J'} = g_{J''}\), as required in Sublem. 6.6.

We are back in Step d. For each \(Y'_I \xrightarrow{f'_I} Y\) in \(\mathcal{F} \downarrow Y\), there exists \(J_I \in J\) such that \(Y_{J_I} = Y'_I\) and \(f_{J_I} = f'_I\) (one can take the initial object 0, which is FP, as \(Q_{J_I}\)). Using such \(J_I\) we obtain an arrow

\[
[g_{J_I}]_{I \in I} : \text{Colim}_{I \in I} Y'_I \rightarrow Z .
\]

That this is a mediating arrow, i.e. that the diagram

\[
\begin{array}{c}
Y_J \\
\downarrow g_J \\
\text{Colim}_{I \in I} Y'_I \xleftarrow{[g_{J_I}]_{I \in I}} Z
\end{array}
\]

commutes, is precisely the content of Sublem. 6.6. Uniqueness of a mediating arrow is easy, too. This proves \(\text{Colim}_{J \in J} Y_J \cong \text{Colim}_{I \in I} Y'_I\).

**Step e.** By Step d we obtain \(Y = \text{Colim}_{J \in J} Y_J\). In view of Lem. C.9, we are done if we show that \(Q = \bigvee_{J \in J} \coprod_{f_J} Q_J\) is a cocone.

One direction \(Q \geq \bigvee_{J \in J} \coprod_{f_J} Q_J\) is easy: since \(f_J\) in (24) is an arrow in \(\text{Sub}(C)\) we have \(Q_J \leq f_J Q\), that is, \(\coprod_{f_J} Q_J \leq Q\), for each \(J \in J\).

To prove the other direction \((Q \leq \bigvee_{J \in J} \coprod_{f_J} Q_J)\), let \((Q_K)_{K \in K}\) be the canonical diagram for \(Q\) with respect to \(F\) (i.e. \(K = (\mathcal{F} \downarrow Q)\)), with the canonical cocone \((Q_K \xrightarrow{c_K} Q)_{K \in K}\).

Then \(Q = \text{Colim}_{K \in K} Q_K\) since \(C\) is LFP; furthermore, much like the proof of Sublem. 6.5, we can show that \(\text{Colim}_{K \in K} Q_K \cong \bigvee_{K \in K} \coprod_{c_K} Q_K\). Hence it suffices to show

\[
\coprod_{c_K} Q_K \leq \bigvee_{J \in J} \coprod_{f_J} Q_J \quad \text{in } \text{Sub}(Y), \text{for each } K \in K .
\]  

(25)

It is easy to see (using (34)) that \(n \circ c_K\) is an arrow

\[
(Q_K \xrightarrow{id} Q_K) \xrightarrow{n \circ c_K} (Q \xrightarrow{n} Y)
\]

in \(\text{Sub}(\mathcal{C})\). Since \(Q_K \in \mathcal{F}\), this arrow \(n \circ c_K\) is an object of the index category \(J = (\mathcal{F}_{\text{Sub}(\mathcal{C})} \downarrow n)\). This yields

\[
\coprod_{n \circ c_K} Q_K \leq \bigvee_{J \in J} \coprod_{f_J} Q_J .
\]  

(26)

Now the following diagram shows that \(\coprod_{n \circ c_K} Q_K = \coprod_{c_K} Q_K\) as a subobject of \(Y\), via
Therefore (26) proves (25). This concludes the proof.

It follows from Lem. 6.3, 6.4, and Cor. 6.2 that the internal logic of a topos that is LFP is finitely determined.

**Corollary 6.7.** Let $\mathcal{C}$ be LFP and at the same time a topos (or more generally an LCCC). Then the subobject fibration $\text{Sub}(\mathcal{C})$ is finitely determined.

### 6.2. Family Fibrations

We turn to the family fibration $\text{Fam}(\Omega) \downarrow \text{Sets}$ over a poset $\Omega$ (see Appendix C). In the following result we use the notion of \textit{finitely generated object}. It is a relaxation of FP object; see Def. B.7.

**Lemma 6.8.** Let $\Omega$ be an \textit{algebraic lattice}, i.e. a complete lattice in which each element is a join of compact elements. (Equivalently, $\Omega$ is LFP when thought of as a category.) Then the set

$$F_{\text{Fam}(\Omega)} := \{ f : X \to \Omega \mid X \text{ is finite, and for each } x \in X, f(x) \text{ is compact in } \Omega \}$$

(27)

consists of finitely generated objects and is dense in $\text{Fam}(\Omega)$. Noting that $f \in F_{\text{Fam}(\Omega)}$ is above a finite set $X$, by Lem. 6.1, $\text{Fam}(\Omega) \downarrow \text{Sets}$ is finitely determined.

**Proof.** \textbf{Step a.} Let us first see that the fibration $\text{Fam}(\Omega) \downarrow \text{Sets}$ has fiberwise limits and colimits and coproducts $\coprod$ between fibers. The former follows from $\Omega$ being a complete lattice; the latter is shown from (Jacobs, 1999, Lem. 1.9.5). In view of Lem. C.9, it follows that the total category $\text{Fam}(\Omega)$ is cocomplete.

\textbf{Step b.} Before going on we prove the following.

**Sublemma 6.9.** An arrow in $\text{Fam}(\Omega)$ is a mono if and only if its underlying function is a mono in $\text{Sets}$.

**Proof.** (of Sublem. 6.9) The “if” part is obvious. For the “only if” part, let $(X \xrightarrow{f} \Omega)$ $\xrightarrow{m}$ $(Y \xrightarrow{g} \Omega)$ be a monic arrow in $\text{Fam}(\Omega)$, and $k, l : U \to X$ be arrows in $\text{Sets}$ such that $m \circ k = m \circ l$. This induces the following situation in $\text{Fam}(\Omega)$:

$$(\perp : U \to \Omega) \xrightarrow{k} f : X \to \Omega \xrightarrow{m} (g : Y \to \Omega) ,$$
where ⊥: U → Ω is the constant function to the least element ⊥ ∈ Ω. Therefore k = l; this proves Sublem. 6.9.

**Step c.** We prove that each \((X \xrightarrow{f} \Omega) \in \mathbb{P}_{\text{Fam}(\Omega)}\) is FG (Def. B.7) in \(\text{Fam}(\Omega)\). Let \(\big((Y_I \xrightarrow{g_I} \Omega) \xrightarrow{h_I} (Y \xrightarrow{g} \Omega)\big)_{I \in \mathcal{I}}\) be a colimiting cocone in \(\text{Fam}(\Omega)\) from a directed diagram \(\mathcal{I}\) such that all the arrows in \(\mathcal{I}\) are monos; and \((X \xrightarrow{f} \Omega) \xrightarrow{l} (Y \xrightarrow{g} \Omega)\) be an arrow in \(\text{Fam}(\Omega)\). We aim at showing that \(k\) factors through some \(h_I\).

By Lem. C.9 we obtain that \(Y = \text{Colim}_{I \in \mathcal{I}} Y_I\); and that \(g(y) = \big(\bigvee_{I \in \mathcal{I}} (\bigvee_{y' \in h_I^{-1}(y)} g_I(y'))\big)\) for each \(y\in Y\).

\(\text{(28)}\)

The first equality is by Lem. C.9; the second is because the order in the fiber \(\text{Fam}(\Omega)_Y = \Omega^Y\) is pointwise; and the third is by the concrete description (Jacobs, 1999, Lem. 1.9.5) of \(\bigvee\) in \(\downarrow \text{Sets}\).

We observe that each \(h_I\) is a mono in \(\text{Fam}(\Omega)\). To see it, \((Y_I \xrightarrow{h_I} Y)_{I \in \mathcal{I}}\) is a colimiting cocone in \(\text{Sets}\) from a directed diagram of monos; since \(\text{Sets}\) is LFP, we can use (Adámek and Rosický, 1994, Prop. 1.62) to show that each \(h_I\) is a mono in \(\text{Sets}\); and then we use Sublem. 6.9.

Now we have \(f(x) \leq g(k(x)) = \bigvee_{I \in \mathcal{I}} \left( \bigvee_{y' \in h_I^{-1}(k(x))} g_I(y') \right)\);

here the first inequality is because \(k\) is an \(\text{Fam}(\Omega)\)-arrow; and the second equality is from (28). By the assumptions that \(f(x)\) is compact and that \(X\) is finite, there exists \(I_0 \in \mathcal{I}\) such that \(f(x) \leq \bigvee_{y' \in h_{I_0}^{-1}(k(x))} g_{I_0}(y')\) for each \(x \in X\) (recall that \(\mathcal{I}\) is filtered).

Furthermore, since \(X \xrightarrow{l} Y = \text{Colim}_{I \in \mathcal{I}} Y_I\) is an arrow from an FP object (in \(\text{Sets}\)) to a directed colimit, it factors through some \(h_{I_1}\):

\[
\begin{array}{ccc}
X & \xrightarrow{l_{I_1}} & Y_{I_1} \\
\downarrow k & & \downarrow h_{I_1} \\
& Y &
\end{array}
\]

By choosing suitable \(I_2\) such that \(I_0, I_1 \leq I_2\), we have \(f(x) \leq \bigvee_{y' \in h_{I_2}^{-1}(k(x))} g_{I_2}(y') = g_{I_2}(l_{I_2}(x))\) for each \(x \in X\);

here the last equality holds since \(h_I\) is an injection and \(h_I(l_I(x)) = k(x)\). This proves that \(l_{I_2}\) is a \(\text{Fam}(\Omega)\)-arrow \((X \xrightarrow{f} \Omega) \to (Y_{I_2} \xrightarrow{g_{I_2}} \Omega)\), hence \(k = h_{I_2} \circ l_{I_2}\) in \(\text{Fam}(\Omega)\). This concludes Step c.

**Step d.** The collection \(\mathbb{P}_{\text{Fam}(\Omega)}\) is obviously small.
Step e. We are done if we prove that every object $P \in \text{Fam}(\Omega)$ is a directed colimit of its subobjects from $\mathcal{F}_{\text{Fam}(\Omega)}$. This easily follows from the fact that the same is true in $\text{Sets}$ (obvious) and in $\Omega$ (being an algebraic lattice).

It is known that the existence of a dense set of FG objects (like $\mathcal{F}_{\text{Fam}(\Omega)}$ in Lem. 6.8) ensures the category to be locally $\lambda$-presentable. This is however for some regular cardinal $\lambda$ that is possibly bigger than $\omega$. See (Adámek and Rosický, 1994, Thm. 1.70).

**Corollary 6.10.** Let $\Omega$ be an algebraic lattice. Then the total category $\text{Fam}(\Omega)$ of $\downarrow_{\text{Sets}}$ is locally presentable.

### 6.3. Presheaf Categories

Presheaf categories are well-known examples of LFP categories. See (Adámek and Rosický, 1994).

**Example 6.11 (Presheaf categories).** Let $\mathcal{A}$ be small. The presheaf category $\text{Sets}^{\mathcal{A}}$ is LFP; the set $\mathcal{F}$ of finite colimits of representable presheaves $\mathbf{y}A$, where $\mathbf{y}A = \mathbf{A}(\mathbf{1}, -)$, satisfies the conditions of Def. 3.1. Indeed, any presheaf $X$ is a filtered colimit of objects in $\mathcal{F}$ since $X$ is a colimit (that is not necessarily filtered) of representable presheaves (Lem. 6.15).

For the subobject fibration of a presheaf category $\text{Sets}^{\mathcal{A}}$, Cond. 4 and 3 in Def. 3.3 (for $X \in \mathcal{F}$) reduce to the representable case $X = \mathbf{y}A$.

**Lemma 6.12.** The subobject fibration $\downarrow_{\text{Sets}^{\mathcal{A}}}$ is well-founded if and only if for all $A \in \mathcal{A}$ the poset $\text{Sub}(\mathbf{y}A)$ has no strictly descending chain. The subobject fibration is co-well-founded if and only if for all $A \in \mathcal{A}$ the poset $\text{Sub}(\mathbf{y}A)$ has no strictly ascending chain.

**Sublemma 6.13.** Let $(X_I)_I$ be a finite diagram in $\text{Sets}^{\mathcal{A}}$. If for each $I$ the poset $\text{Sub}(X_I)$ has no strictly descending chain, then so does $\text{Sub}(\text{Colim}_I X_I)$. If for each $I$ the poset $\text{Sub}(X_I)$ has no strictly ascending chain, then so does $\text{Sub}(\text{Colim}_I X_I)$.

**Proof.** (Of Sublem. 6.13) We rely on a presentation of colimits by coproducts and coequalizers. In a topos (hence a regular category) $\text{Sets}^{\mathcal{A}}$ coproducts are disjoint (see e.g. (Jacobs, 1999, Exercise 4.5.1)); thus we have an isomorphism of posets

\[
\text{Sub}(X_1 + \cdots + X_n) \cong \text{Sub}(X_1) \times \cdots \times \text{Sub}(X_n) .
\]

Let $X \xrightarrow{\sim} Y \xrightarrow{e} Z$ be a coequalizer in $\text{Sets}^{\mathcal{A}}$. The correspondence $e^*: \text{Sub}(Z) \rightarrow \text{Sub}(Y)$ is easily seen to be injective. Indeed, assume $P \not\cong P'$ in $\text{Sub}(Z)$; then $PA \not\cong P'A$ in $\text{Sets}$ for some $A \in \mathcal{A}$, and since $e_A: \mathbf{Y}A \rightarrow ZA$ is surjective, we have

\[
(e^*P)A = e_A^{-1}(PA) \not\cong e_A^{-1}(P'A) = (e^*P')A .
\]

Therefore if $\text{Sub}(Z)$ has a strictly descending or ascending chain, $\text{Sub}(Y)$ has a strictly descending or ascending chain respectively. This concludes the proof of the sublemma. \qed
Proof. (Of Lem. 6.12) By Example 6.11, Cor. 6.7, and Sublem. 6.13.

The previous lemma reduces the size problem of the fibration \(\text{Sub}^{\text{Sets}^A}\) to that of \(\text{Sub}(yA)\). In calculating \(\text{Sub}(yA)\), we will be using the following well-known characterization of presheaves as colimits of representables.

**Definition 6.14.** Let \(A\) be a small category and \(P: A \to \text{Sets}\) be a functor. The category of elements of \(P\), which is denoted by \(\int P\), consists of objects that are pairs \((A, p)\in A\times P\) and arrows

\[
(\int P)((A, p), (B, q)) = \{f: A \to B \mid P(f)(p) = q\}.
\]

**Lemma 6.15.** Any presheaf \(P \in \text{Sets}^A\) is canonically isomorphic to the colimit of representable functors indexed by the category of elements:

\[
P \sim = \text{Colim}_{(A,p) \in \int P} yA.
\]

**Proof.** For each object \((A, p) \in \int P\), an arrow \(yA \to P\) is induced by \((yA)B = \text{Hom}(A, B) \ni g \mapsto P(g)(p) \in P(B)\). It is not difficult to see that these arrows \(yA \to P\) are natural in \((A, p)\in \int P\) and form a colimiting cocone. See e.g. (Adámek and Rosický, 1994, Prop. 1.45) for details.

Prop. 6.17 (presented later) will be our principal tool for calculating \(\text{Sub}(yA)\). The proposition is inspired by the following cocompletion results (Lem. 6.16), which will not be themselves used in our subsequent technical developments.

**Lemma 6.16.** Let \(A\) be a small category.

1. The category \(\text{Sets}^A\) of presheaves is a free cocompletion of the category \(A^{\text{op}}\) (with the unit \(y: A^{\text{op}} \to \text{Sets}^A\)), that is, for a functor \(F: A^{\text{op}} \to C\) to a cocomplete category \(C\) there uniquely (up to natural isomorphisms) exists a cocontinuous functor \(G: \text{Sets}^A \to C\) such that \(F \cong G \circ y\).
2. Let \(P \in \text{Sets}^A\) be a presheaf. There exists an equivalence of categories \(\text{Sets}^A/P \cong \text{Sets}^{fP}\). Hence, the slice category \(\text{Sets}^A/P\) is a free cocompletion of the category \((fP)^{\text{op}}\).
3. Let \(A \in A\) be an object. The category \(\text{Sets}^A/(yA)\) is equivalent to the category \(\text{Sets}^{A/A}\). Hence, the slice category \(\text{Sets}^A/(yA)\) is a free cocompletion of the category \((A/A)^{\text{op}} = A^{\text{op}}/A\).

**Proof.** The item 1 is well-known: the functor \(G\) is given by \(GP = \text{Colim}_{(A,p) \in fP} FA\). In particular, when we take \(y: A^{\text{op}} \to \text{Sets}^A\) as \(F\), we obtain \(G\) that is naturally isomorphic to id: \(\text{Sets}^A \to \text{Sets}^A\). This generalizes Lem. 6.15.

The item 2—with a strong fibrational flavor, via the Grothendieck construction—is found e.g. in (Mac Lane and Moerdijk, 1992, Exercise III.8.(a)). The equivalence is given explicitly by

\[
\begin{align*}
\text{Sets}^A/P & \longrightarrow \text{Sets}^{fP} \\
(Q \xrightarrow{\alpha} P) & \longmapsto [(A, p) \mapsto (\alpha_A)^{-1}(\{p\})] \\
\text{Sets}^{fP} & \longrightarrow \text{Sets}^A/P \\
R & \longmapsto [A \mapsto \bigsqcup_{p \in P} R(A, p)]
\end{align*}
\]
where, in the last entry, we only presented a presheaf in $\text{Sets}^\mathcal{A}$ (an arrow to $P$ is given obviously by a projection).

The item 3 is obtained from the item 2 and the fact that $f(yA) = A/\mathcal{A}$ (an easy observation).

**Proposition 6.17.**

1. Let $\mathcal{A}$ be small. For any $A \in \mathcal{A}$, the subset

$$\{\text{Im}(yB \xrightarrow{f} yA) \mid B \in \mathcal{A}, f : A \to B\} \subseteq \text{Sub}(yA)$$

is dense as a full subcategory, that is, for any subpresheaf $Q \to yA$ there canonically exists a family $(f_i : A \to B_i)_I$, such that $Q = \bigcup_I \text{Im}(yf_i)$.

2. Furthermore, assume that every arrow $f$ with domain $A \in \mathcal{A}$ factors as $f = m \circ e$ with an epi $e$ and a split mono $m$. Then, (the image of) the canonical embedding $\text{Quot}(A) \to \text{Sub}(yA)$ is dense. Here $\text{Quot}(A)$ denotes the poset of quotient objects of $A$.

**Proof.** A detailed proof is given in Appendix D.

**Corollary 6.18.** If the following condition 1 holds for each $A \in \mathcal{A}$, then the fibration $\text{Sub}(\text{Sets}^\mathcal{A}) \downarrow \text{Sets}^\mathcal{A}$ is both well-founded and co-well-founded.

1. The subset $\{\text{Im}(yf) \mid B \in \mathcal{A}, f : B \to A\} \subseteq \text{Sub}(yA)$ is finite.

Furthermore, for each $A \in \mathcal{A}$, the following condition 2 implies the condition 1 above.

2. Any arrow $f$ with domain $A \in \mathcal{A}$ factors as $f = m \circ e$ with an epi $e$ and a split mono $m$, and moreover, $\text{Quot}(A)$ is a finite set.

**Proof.** By Lem. 6.12, it is enough to show that for each $A \in \mathcal{A}$ the poset $\text{Sub}(yA)$ is finite.

Assume that $A \in \mathcal{A}$ satisfies the condition 1: the subset $\{\text{Im}(yf) \mid B \in \mathcal{A}, f : B \to A\} \subseteq \text{Sub}(yA)$ is finite. By Prop. 6.17.1, we have $\text{Sub}(yA) = \{\bigcup_I \text{Im}(yf_i) \mid (B_i \in \mathcal{A}, f_i : B_i \to A)\}I$, which is also finite.

That the condition 2 implies 1 follows from Prop. 6.17.2.

To determine whether $\text{Im}(yf) = \text{Im}(yg)$ holds for arrows $f$ and $g$ with the same domain, the following lemma is useful.

**Lemma 6.19.** The inclusion relation $\subseteq$ on $\{\text{Im}(yf) \in \text{Sub}(yA) \mid B \in \mathcal{A}, f : A \to B\}$ is the partial order induced by the preorder $\subseteq$ on $\{f \mid B \in \mathcal{A}, f : A \to B\}$. The latter is defined by:

$$(f : A \to B) \subseteq (g : A \to C) \text{ if and only if } f = h \circ g \text{ for some } h : C \to B.$$ 

**Proof.** Let $f : A \to B$, $g : A \to C$ be arrows in $\mathcal{A}$. We first observe that

$$(\text{Im}(yf))D = \text{Im}((yB)D \xrightarrow{yf} yA)D$$

$$= \{(yf)D(k) \mid k \in (yB)D\}$$

$$= \{k \circ f : A \to D \mid k : B \to D\} \quad (29)$$
Coinductive Predicates and Final Sequences in a Fibration

31

for $D \in \mathcal{A}$.

Assume that $\text{Im}(yf) \subseteq \text{Im}(yg)$ in $\text{Sets}^\mathcal{A}$. In particular, it holds $(\text{Im}(yf))B \subseteq (\text{Im}(yg))B$ in $\text{Sets}$. We have $f = \text{id}_B \circ f \in (\text{Im}(yf))B$ by (29), hence $f \in (\text{Im}(yg))B$. Thus, there exists $h: C \to B$ such that $f = h \circ g$, which is the definition of $f \preceq g$.

Conversely, assume that $f = h \circ g$ for some $h: C \to B$. For any $D \in \mathcal{A}$, we have

$$(\text{Im}(yf))D = \{k \circ h \circ g: A \to D \mid k: B \to D\}$$

by (29)

$\subseteq \{k': A \to D \mid k': C \to D\} = (\text{Im}(yg))D$ by (29).

Therefore $\text{Im}(yf) \subseteq \text{Im}(yg)$.

7. Concrete Examples

Example 7.1 (Pred). The fibration $\text{Pred} \downarrow \text{Sets}$ for the conventional setting of classical logic is easily seen to be well-founded and co-well-founded. In particular, $\text{Pred}_X \cong \mathcal{P}X$ is finite if $X$ is FP (i.e., finite). Therefore to any finitary $F$ and any predicate lifting $\varphi$, the results in §3 apply.

The (interpretations of the) formulas in $\mathcal{R} \nu$ (see Example 3.9) are examples of coinductive predicates in $\text{Pred} \downarrow \text{Sets}$. Besides them, the study of coalgebraic modal logic has identified many predicate liftings for many functors $F$ (probabilistic systems, neighborhood frames, strategy frames, weighted systems, etc.; see e.g., (Cîrstea et al., 2011) and the references therein). These “modalities” all define coinductive predicates, to which the results in §3 may apply.

Example 7.2 (Rel). The fibration $\text{Rel} \downarrow \text{Sets}$ can be introduced from $\text{Pred} \downarrow \text{Sets}$ via change-of-base; concretely, an object of $\text{Rel} \downarrow \text{Sets}$ is a pair $(X,R)$ of a set $X$ and a relation $R \subseteq X \times X$; an arrow $f: (X,R) \to (Y,S)$ is a function $f: X \to Y$ such that $xRx'$ implies $f(x)Sf(x')$. See (Jacobs, 1999, p. 14).

This fibration, similarly to $\text{Pred} \downarrow \text{Sets}$, is easily seen to be well-founded and co-well-founded; therefore to any finitary $F$ the results in §3 apply. A predicate lifting $\varphi$ along $\text{Rel} \downarrow \text{Sets}$ is more commonly called a relation lifting (Hermida and Jacobs, 1998); by choosing suitable $\varphi$ for given $F$ (a “sufficiently comprehensive” one) like in (Hermida and Jacobs, 1998), a $\varphi$-invariant is precisely an $F$-bisimulation relation (in the coalgebraic sense), and the $\varphi$-coinductive predicate is $F$-bisimilarity. We expect that the $\omega$-behavioral bound in Thm. 3.8 can be used to bound execution of bisimilarity checking algorithms by partition refinement (for many different functors $F$).

In the following example, one can think of $\Omega$ as a Heyting algebra, and then the underlying logic becomes constructive.

Example 7.3 (Fam($\Omega$)). Let $\Omega$ be an algebraic lattice that has no strictly descending ($\omega^\text{op}$-)chains. Then the family fibration $\text{Fam}(\Omega) \downarrow \text{Sets}$ is well-founded (see Lem. 6.8). Therefore
to any finitary $F$ the results in §3 apply. It is not hard to interpret the language $\mathbb{R}_\nu$ in this setting, by defining predicate liftings similar to (7). This gives examples of coinductive predicates in $\text{Fam}(\Omega) \downarrow \text{Sets}$.

Similarly, fibrations $\text{Fam}(\Omega^o) \downarrow \text{Sets}$ are co-well-founded for algebraic lattices $\Omega$ by Lem. 5.6, because the fibrations are fiberwise opposite of well-founded fibrations $\text{Fam}(\Omega) \downarrow \text{Sets}$.

7.1. Presheaf Examples

Let $\mathbf{F}$ be the category of natural numbers as finite sets (i.e. $n = \{0, 1, \ldots, n - 1\}$) and all functions between them; $\mathbf{F}_+$ be its full subcategory of nonzero natural numbers; and $\mathbf{I}$ be the category of natural numbers and injective functions. Coalgebras in the presheaf categories $\text{Sets}^\mathbf{F}$, $\text{Sets}^\mathbf{F}_+$ and $\text{Sets}^\mathbf{I}$ are commonly used for modeling processes in various name-passing calculi. For the $\pi$-calculus $\text{Sets}^\mathbf{I}$ has been found appropriate (see e.g. (Fiore and Turi, 2001; Fiore and Staton, 2006)); while for the fusion calculus we do need non-injective functions in $\mathbf{F}$ or $\mathbf{F}_+$ (see (Miculan, 2008; Staton, 2011)).

Inspired by (Klin, 2007), we are interested in coinductive predicates for such processes. They are naturally modeled in the subobject fibration of a presheaf category. Here we find a distinction: the subobject fibrations of both $\text{Sets}^\mathbf{F}$ and $\text{Sets}^\mathbf{F}_+$ are well-founded and co-well-founded; but that of $\text{Sets}^\mathbf{I}$ is not well-founded (it is co-well-founded). In view of Lem. 6.12, the only condition to check is Cond. 4 or 5 for $X = yA$.

Example 7.4 $(\text{Sub}(\text{Sets}^\mathbf{F}), \text{Sub}(\text{Sets}^\mathbf{F}_+))$. The subobject fibration $\text{Sub}(\text{Sets}^\mathbf{F}_+) \downarrow \text{Sets}^\mathbf{F}_+$ is well-founded and co-well-founded: this is shown by that the second condition of Cor. 6.18 holds for any $A \in \mathbf{F}_+$. An important fact here is that in $\mathbf{F}$ (or in $\text{Sets}$) a mono with a nonempty domain splits, and thus every mono in $\mathbf{F}_+$ is a split mono.

The subobject fibration $\text{Sub}(\text{Sets}^\mathbf{F}) \downarrow \text{Sets}^\mathbf{F}$ is well-founded and co-well-founded, too. To show that $\text{Sub}(y0)$ is finite, we appeal directly to the first condition of Cor. 6.18: we observe by Lem. 6.19 that the set $\{\text{Im}(yf) \mid n \in \mathbf{F}, f : 0 \to n\}$ is equal to the two-element set $\{\text{Im}(y(0 \xrightarrow{n} 0)), \text{Im}(y(0 \xrightarrow{1} 1))\}$ since $0 \xrightarrow{\delta} n$ and $0 \xrightarrow{1} m$ factor through each other, for each $n, m \geq 1$.

We turn to functors $F$ and $\varphi$. In modeling processes of name-passing calculi as coalgebras in these categories, one typically uses endofunctors $F$ that are constructed from the following building blocks. Let $\mathbf{N} \in \{\mathbf{F}, \mathbf{F}_+, \mathbf{I}\}$.

— Constant functors, binary sum $+$, binary product $\times$, and exponentials $(\_)^X$. These are much like for polynomial functors on $\text{Sets}$. An important example of the first is the name presheaf $\mathcal{N} = \text{Hom}(1, \_)$ in $\text{Sets}^\mathbf{N}$.

— The abstraction functor $\delta : \text{Sets}^\mathbf{N} \to \text{Sets}^\mathbf{N}$ given by $\delta x = X(\_ + 1)$.

— The free semilattice functor $\mathcal{P}_f$ for finite branching. This captures Kuratowski finiteness and suitable in $\text{Sets}^\mathbf{I}$. See e.g. (Fiore and Turi, 2001; Staton, 2011).
— In Sets\textsuperscript{F} and Sets\textsuperscript{F+}, another choice of a “finite powerset functor” \(\tilde{K}\) is more appropriate. See (Miculan, 2008); also (Staton, 2011, p. 4).

All such functors are known to be finitary (see e.g. (Miculan, 2008)).

Coinductive predicates in this setting can be introduced much like \(R_\nu\) in Example 2.4 (note that Sets\textsuperscript{N} is a topos for \(N \in \{F, F+, I\}\), for properties like deadlock freedom. Such a language can be extended further through the modalities proposed in (Klin, 2007): they correspond to constructions specific to presheaves and include the modality \(\langle a(b) \rangle\) for a binding “input” operation. More examples will be worked out in our future paper.

\textbf{Example 7.5 (Sub(Sets\textsuperscript{\omega}), Sub(Sets\textsuperscript{I})).} Consider the presheaf category Sets\textsuperscript{\omega} over the ordinal \(\omega\) as a poset. The fibration \(\text{Sub}\textsuperscript{\omega} \downarrow \text{Sets} \textsuperscript{\omega}\) is finitely determined but not well-founded. It fails to satisfy Cond. 4 in Def. 3.3: let \(P_n: \omega \to \text{Sets} \textsuperscript{\omega}\) be the family of presheaves defined by
\[
P_n(m) := \begin{cases} 0 & \text{if } m < n; \\ 1 & \text{if } n \leq m \end{cases}
\]
for each \(n \in \omega\). Then \(P_0 > P_1 > \cdots\) is a strictly descending chain in Sub(y0). The same counterexample works for Sub(Sets\textsuperscript{I}).

In contrast, the fibrations \(\text{Sub} \textsuperscript{\omega} \downarrow \text{Sets} \textsuperscript{\omega}\) and \(\text{Sub} \textsuperscript{I} \downarrow \text{Sets} \textsuperscript{I}\) are co-well-founded, by Lem. 6.12 and the following lemma.

\textbf{Lemma 7.6.} For \(A \in \{\omega, I\}\) and for any \(n \in A\), the poset Sub\textsuperscript{A} is isomorphic to the opposite of the ordinal \(\omega + 1 = \omega \cup \{\omega\}\). Hence Sub\textsuperscript{A} has no strictly increasing chain.

\textit{Proof.} Firstly we shall invoke Lem. 6.19. Let \(f: n \to m\) be an arrow in \(A\). Note that the existence of the arrow \(f\) induces \(m \geq n\) as natural numbers. For an arrow \(g: n \to m'\) in \(A\), it is easy to see that \(f\) factors through \(g\) if and only if \(m' \leq m\). In particular, arrows \(f, f': n \Rightarrow m\) factor through each other; therefore we may denote by \(\text{Im}(ym)\) the image \(\text{Im}(ym) \subseteq \text{Sub}(yn)\) for \(m \geq m'\).

Therefore there exists an isomorphism of posets
\[
J : \omega^\text{op} \cong \{\text{Im}(ym) \mid m \geq n\} = \{\text{Im}(ym)^{\downarrow} \mid y n \in \text{Sub}(yn) \mid m \in A, f : n \to m\}
\]
defined by \(I(k) = \text{Im}(y(n + k))\).

We shall induce an isomorphism (the monotone function \(J\) below) between the “co-completion” of both-hand sides of the isomorphism \(I\). Let DSub\textsuperscript{A} be the poset of downward closed subsets of \(\omega^\text{op}\) ordered by inclusion, and \(\omega + 1\) be the ordinal. Let \(h: (\omega + 1)^\text{op} \to \text{DSub}(\omega^\text{op})\) be a function such that
\[
h(k') = \begin{cases} \downarrow k & \text{if } k' = k \in \omega; \\ 0 & \text{if } k' = \omega \end{cases}
\]
for \(k' \in \omega + 1\), where \(\downarrow k = \{k, k + 1, \ldots\}\) is the downward closure of \(\{k\} \subseteq \omega^\text{op}\). It is easy to see that \(h\) becomes an isomorphism of posets. Since the poset Sub\textsuperscript{A} is cocomplete,
the isomorphism $I$ induces the diagram

$$
\begin{array}{ccc}
\omega^{op} & \xrightarrow{I} & \{\text{Im}(ym \xrightarrow{f} yn) \mid m \in A, f : n \to m\} \\
\downarrow \circ & & \downarrow \\
\text{DSub}(\omega^{op}) & \xrightarrow{I} & \text{Sub}(yn)
\end{array}
$$

in Posets, where $i$ is the canonical inclusion, and $J(S) = \bigcup_{k \in S} I(k)$ is the union of the images under the isomorphism. It is enough to show that $J$ is also an isomorphism of posets.

On the one hand, the inclusion $i$ is dense as a full subcategory by Prop. 6.17.1, that is, the monotone function $J$ is surjective. On the other hand, the nullary union $J(\emptyset) = 0$ in Sub$(yn)$ is strictly less than any other image $J(\downarrow k) = I(k) = \text{Im}(y(n + k))$ for $k \in \omega$. Hence the monotone function $J : \text{DSub}(\omega^{op}) \to \text{Sub}(yn)$ is an embedding (i.e. a monotone injection that reflects the order) that extends the embedding $i \circ I : \omega^{op} \to \text{Sub}(yn)$. Therefore, the monotone function $J$ is a surjective embedding, that is, an isomorphism of posets.

In contrast to Sets$^{\omega}$, the subobject fibration for Sets$^{\omega^{op}}$ is well-founded and co-well-founded by Cor. 6.18. Indeed, arrows $f : n \to m$ in $\omega^{op}$ has an (Epi, SplitMono)-factorization $n \twoheadrightarrow m \rightarrowtail m$, and Quot$_{\omega^{op}}(n) = \{n, n - 1, \ldots, 0\}$ is a finite set.

Remark 7.7. Well-foundedness fails in Sub$(\text{Sets}^{\omega})$, Sub$(\text{Sets}^{I})$, and in Fam$(\Omega)$ for $\Omega$ that does have a strictly descending $\omega^{op}$-chain. This means the logics modeled by the fibrations are inherently “big.” Still, extensions of our results in §3 are possible from finitary (i.e. $\omega$-presentable) to the $\lambda$-presentable setting for bigger $\lambda$, so that they apply to the (current) nonexamples.

8. Conclusions and Future Work

We have investigated a mathematical theory of coinductive (and inductive) predicates over coinductive datatypes, formalized categorically using coalgebras and fibrations. Our technical results are about iterative constructions of coinductive predicates; they are stated also in abstract categorical terms, using the language of locally presentable categories.

In this paper we focused on purely coinductive predicates and purely inductive ones. However in system verification their combination is very commonly used. Such mixture of induction and coinduction is studied fibrationally in (Hensel and Jacobs, 1997), but over mixed inductive and coinductive data types, and not over a coalgebra.

Search for useful coinduction proof principles is an active research topic (see e.g. (Bonchi and Pous, 2013; Hur et al., 2013)). We are interested in the questions of whether these principles are sound in a general fibrational setting, and what novel proof principles a fibrational view can lead to. In fact the well-known technique of coinduction up-to has been formulated in fibrational terms (Bonchi et al., 2014) and revealed exciting new applications like nominal automata.

Coalgebraic modal logic is more and more often introduced based on a Stone-like


duality (see e.g. (Klin, 2007)). Fibrational presentation of such dualities will combine the benefits of duality-based modal logics and the current results. We are also interested in the relationship to coalgebraic infinite traces (Jacobs, 2004; Cîrstea, 2011).

Kozen’s metric coinduction (Kozen and Ruozzi, 2009) is a construction of coinductive predicates by the Banach fixed point theorem and is an alternative to the current paper’s order-theoretic one. Its fibrational formulation is an interesting future topic.

Practical applications of our categorical behavioral bounds shall be pursued, too. Our results’ precursor—the bounds for the final sequences in Sets (Worrell, 2005; Adámek, 2003)—have been used to bound execution of some algorithms e.g. for state minimization (Adámek et al., 2012; Ferrari et al., 2002; Ferrari et al., 2005). We aim at similar use. Finally, games are an extremely useful tool in fixed point logics (also in their coalgebraic generalization, see (Venema, 2006; Cîrstea and Sadrzadeh, 2008; Cîrstea et al., 2009); also (Kupke, 2007)). We plan to investigate the use of games in the current (even more general) fibrational setting.

Acknowledgments

Special Thanks are due to Bart Jacobs and Claudio Hermida for their inspiring work, discussions, comments and encouragements. Besides, thanks are due to Kazuyuki Asada, Keisuke Nakano, Keiko Nakata, Ana Sokolova, and the participants of Dagstuhl Seminar 12411 “Coalgebraic Logics” (including Samson Abramsky, Vincenzo Ciancia, Corina Cîrstea, Ernst-Erich Doberkat, Clemens Kupke, Alexander Kurz, and Yde Venema) for useful discussions. We are grateful to the anonymous referees of the earlier version (Hassuo et al., 2013) for their careful reading and useful suggestions, too. The authors are supported by Grants-in-Aid for Young Scientists (A) No. 24680001 and Grants-in-Aid for Challenging Exploratory Research No. 23654033, JSPS, and by Aihara Innovative Mathematical Modeling Project, FIRST Program, JSPS/CSTP.

References


Coinductive Predicates and Final Sequences in a Fibration


Appendix A. Theory of Coalgebra

Given a category $\mathcal{C}$ and an endofunctor $F : \mathcal{C} \to \mathcal{C}$, an $F$-coalgebra is a pair of $X \in \mathcal{C}$ and an arrow $c : X \to FX$ (we shall denote a coalgebra simply by $X \xrightarrow{c} FX$). The notion has turned out to be a useful categorical abstraction of state-based dynamic systems. In an $F$-coalgebra $X \xrightarrow{c} FX$, the carrier object $X \in \mathcal{C}$ is understood as a state space; the functor $F$ specifies the behavior type; and the arrow $c$ represents actual dynamics. In the most common setting of $\mathcal{C} = \text{Sets}$, examples of functors $F$ (and the corresponding behavior types) are:
- $A \times (\_)$ for $A$-stream automata;
- $\mathcal{P}(\mathcal{A}) \times \mathcal{P}(\_)$ for Kripke models;
- $\mathcal{P}(\mathcal{A}) \times \mathcal{P}_\omega(\_)$ for finitely branching Kripke models, where $\mathcal{P}_\omega$ is the finite powerset functor;
- $\mathcal{P}(A \times \_)$ for labeled transition systems;
- $\mathcal{D}(A \times \_)$ for generative probabilistic systems;
and so on. See (Rutten, 2000; Jacobs, 2012) for detailed introduction.

In the theory of coalgebra as a categorical theory of (state-based dynamical) systems, the notion of final coalgebra plays a prominent role. A final $F$-coalgebra $Z \xrightarrow{\zeta} FZ$ is one such that, for any $F$-coalgebra $X \xrightarrow{c} FX$, there is a unique morphism of coalgebras from $c$ to $\zeta$.

$$FX \xrightarrow{c} X \xrightarrow{\tau} FZ \xrightarrow{\zeta} Z$$

(30)

Its system-theoretic significance is that: 1 $Z$ is often the collection of “all possible $F$-behaviors”; and 2 the induced arrow $\tau$ assigns, to each state in $X$, its behavior. The “behaviors” here follow a black-box view on systems (it ignores internal states) and often captures the natural notion of “$F$-bisimilarity.”

Therefore a question arises if a final $F$-coalgebra exists. The well-known Lambek lemma (that $\zeta$ is necessarily an iso) prohibits e.g. a final $\mathcal{P}$-coalgebra. What matters here is the size of $F$: when it is suitably bounded, a concrete construction of a final coalgebra is known. It obtains a final coalgebra via a final $F$-sequence (Here $1$ is a final object in $\mathcal{C}$).

$$1 \xleftarrow{1} F1 \xleftarrow{F1} \ldots \xleftarrow{F^{i-1}1} F^i1 \xleftarrow{F^i1} F^{i+1} \ldots$$

(31)

In particular, if $F$ is finitary (a size restriction described later), a final coalgebra arises as a suitable quotient of the limit of the final sequence (4). This construction in $\text{Sets}$ is worked out in (Worrell, 2005); it is further extended to locally presentable categories (those are categories suited for speaking of “size”) with additional assumptions in (Adámek, 2003). The current paper’s goal is to apply this construction also to coinductive predicates.

Appendix B. Locally Finitely Presentable Categories

The theory of coalgebra has been mainly developed in the base category $\mathcal{C} = \text{Sets}$. Exceptions include the category of nominal sets or (pre)sheaf categories (e.g. (Fiore and...
Staton, 2006; Fiore and Staton, 2009)) for name-passing calculi, and Kleisli categories (e.g. (Hasuo et al., 2007; Hasuo, 2010)) for trace semantics and simulation. The current paper follows (Adámek, 2003; Klin, 2007) and finds locally finitely presentable categories a convenient abstract setting. Here we follow (Adámek and Rosický, 1994) and list a minimal set of definitions and results on locally finitely presentable categories.

The following is a categorical formalization of “finiteness” of objects. Examples are finite sets (in \( \text{Sets} \)), and algebras presented by finitely many generators and finitely many equations (in suitable categories of algebras).

**Definition B.1 (Finitely presentable object).** An object \( X \in \mathcal{C} \) is finitely presentable (FP) if the functor \( \mathcal{C}(X, -) : \mathcal{C} \to \text{Sets} \) preserves filtered colimits.

**Definition B.2 (Locally finitely presentable category).** A category \( \mathcal{C} \) is locally finitely presentable (LFP) if it is cocomplete and it has a (small) set \( F \) of FP objects such that every object is a directed colimit of objects in \( F \).

**Lemma B.3.** Let \( \mathcal{C} \) be LFP, with a set \( F \) of FP objects as in Def. 3.1; and \( X \in \mathcal{C} \). The canonical diagram for \( X \) with respect to \( F \)

\[
(F \downarrow X) \xrightarrow{\pi} F \hookrightarrow \mathcal{C}
\]

has \( X \) as its colimit. Here \( \pi \) is the projection.

**Proof.** The proof of (Adámek and Rosický, 1994, Prop. 1.22) yields the claim. \( \square \)

**Lemma B.4.** (Adámek and Rosický, 1994, Cor. 1.28 & Prop. 1.61) Let \( \mathcal{C} \) be LFP.

1. \( \mathcal{C} \) is complete.
2. \( \mathcal{C} \) has (StrongEpi, Mono)- and (Epi, StrongMono)-factorization structures. \( \square \)

The following notion (which is already in Def. B.1) is about the “size” of functors. An intuition (when \( \mathcal{C} = \text{Sets} \)) is: a functor \( F \) is finitary if \( F \)'s action \( FX \) on an arbitrary set \( X \) is determined by its action \( FX' \) on all the finite subsets \( X' \subseteq X \).

**Definition B.5 (Finitary functor).** An endofunctor \( F : \mathcal{C} \to \mathcal{C} \) is finitary if it preserves filtered colimits.

This notion is commonly used to bound the “branching degree” of systems as \( F \)-coalgebras. For example, the finite powerset functor \( P_\omega \) is finitary; the (full) powerset functor \( P \) is not.

There are many LFP categories, among which are \( \text{Sets} \), the category \( \text{Posets} \) of posets and monotone functions, and categories of algebras with finitary operations. See (Adámek and Rosický, 1994) for more examples.

**Example B.6 (Presheaf categories).** Let \( \mathcal{A} \) be a small category. The presheaf category \( \text{Sets}^{\mathcal{A}} \) is LFP: the set

\[
F := \{ \text{finite colimits of representable presheaves } \mathcal{Y}A \},
\]

where \( \mathcal{Y}A = \mathcal{A}(A, -) \), satisfies the conditions of Def. B.1.
Definition B.7 (Finitely generated object). An object $X \in C$ is finitely generated (FG) if the functor $C(X, \_): C \to \text{Sets}$ preserves directed colimits of monos—that is, directed colimits of diagrams in which every (connecting) arrow is a mono.

It is clear that FP implies FG. In algebraic terms, FP objects are algebras presented by finitely many generators and finitely many equations; while for FG objects only a set of generators is required to be finite. The two notions coincide in “non-algebraic” examples such as $\text{Sets}$. See (Ad´amek and Rosick´y, 1994, §1.E).

Appendix C. Fibrations

We follow (Jacobs, 1999), although we focus on the simpler notion of poset fibration.

C.1. Introduction (via Indexed Posets)

This paper’s interest is in coinductive predicates, hence in predicate logic. The most straightforward formalization of predicate is as a subset $P \subseteq X$ of a set (a “universe”) $X$: an element $x \in X$ satisfies $P$ if $x \in P$. Accompanying is the natural notion of entailment: $P$ entails $Q$ if $P \subseteq Q$. This way we obtain the poset $(2^X, \subseteq)$ of predicates over $X$.

However it is not on a single universe $X$ that we consider predicates. For example, in a situation where there are two Kripke models $c = (X, \to, V_X)$, $d = (Y, \to, V_Y)$ and a “homomorphism” $f: X \to Y$, a natural question is if the interpretation of a formula $\nu u.\alpha$ is preserved by $f$. (It is; see Prop. 3.12). Here we are comparing the predicate $[\nu u.\alpha]_c \subseteq X$ with the predicate $[\nu u.\alpha]_d \subseteq Y$ reindexed via $f: X \to Y$. The latter is concretely described as the inverse image

$$f^{-1}(\nu u.\alpha)_d = \{ x \in X \mid f(x) \in \nu u.\alpha \} .$$

Therefore a reindexing structure is also relevant to predicate logic: a function $f: X \to Y$ induces reindexing $f^{-1}: 2^Y \to 2^X$. Additionally, the map $f^{-1}$ is monotone.

To summarize: 1) predicates on a universe $X$ form a poset; 2) a function $f: X \to Y$ between universes induces a monotone reindexing function from the collection of predicates over $X$ to that over $Y$. Such a situation is nicely described as a (contravariant) functor

$$\Phi: C^{\text{op}} \to \text{Posets} ,$$

where $\text{Posets}$ is the category of posets and monotone functions. The functor $\Phi$ assigns, to each “universe” $X \in C$, the poset $\Phi X$ of predicates over $X$. Moreover, $f: X \to Y$ in $C$ induces a reindexing map $\Phi f: \Phi Y \to \Phi X$. This functor $\Phi$ is a special case of an indexed category (Jacobs, 1999, §1.10).

In the current paper, however, we favor an equivalent presentation of such a structure by a fibration, since we find the latter to be more amenable to generalization of structures in ordinary category theory (such as limits). The equivalence between index categories and fibrations are well-known; here we sketch the Grothendieck construction from the
former to the latter. Its idea is to “patch up” the posets \((\Phi X)_{X \in C}\) and form a big category \(P\), as in the following figure.

\[
\Phi X \xleftarrow{\Phi f} \Phi Y
\]

On the right we add some arrows (denoted by \(\rightarrow\)) so that we have an arrow \((\Phi f)(Q) \rightarrow Q\) in \(P\) for each \(Q \in \Phi Y\). (On the left the correspondence \(\rightarrow\) depicts the action of the map \(\Phi f\).) The above diagram in \(P\) should be understood as a Hasse diagram: those arrows which arise from composition are not depicted.

Formally:

**Definition C.1 (The Grothendieck construction).** Given \(\Phi: C^{op} \rightarrow \text{Posets}\), we define the category \(P_{\Phi}\) by

- its object is a pair \((X, P)\) of an object \(X \in C\) and an element \(P\) of the poset \(\Phi X\); and
- its arrow \(f: (X, P) \rightarrow (Y, Q)\) is an arrow \(f: X \rightarrow Y\) in \(C\) such that

\[P \leq (\Phi f)(Q)\]

Here \(\leq\) refers to the order of \(\Phi X\).

Thus arises a category \(P_{\Phi}\) that incorporates: the order structure of each of the posets \((\Phi X)_{X \in C}\); and the reindexing structure by \((\Phi f)_{c-\text{arrow}}\). For fixed \(X \in C\), the objects of the form \((X, P)\) and the arrows \(\text{id}_X\) between them form a subcategory of \(P\). This is denoted by \(P_X\) and called the fiber over \(X\). It is obvious that \(P_X\) is a poset that is isomorphic to \(\Phi X\).

Moreover, there is a canonical projection functor \(p: P \rightarrow C\) that carries \((X, P)\) to \(X\).

**C.2. Formal Definition of (Poset) Fibration**

We axiomatize those structures which arise in the way described above.

**Definition C.2 ((Poset) fibration).** A (poset) fibration \(\xrightarrow{p} C\) consists of two categories \(P, C\) and a functor \(p: P \rightarrow C\), that satisfy the following properties.

- Each fiber \(P_X\) is a poset. Here the fiber \(P_X\) for \(X \in C\) is the subcategory of \(P\) consisting of objects \(P \in P\) such that \(pP = X\) and arrows \(f: P \rightarrow Q\) such that \(pf = \text{id}_X\) (such arrows are said to be vertical).
- Given \(f: X \rightarrow Y\) in \(C\) and \(Q \in P_Y\), there is an object \(f^*Q \in P_X\) and a \(P\)-arrow \(\overline{fQ}: f^*Q \rightarrow Q\) with the following universal property. For any \(P \in P_X\) and \(g: P \rightarrow Q\) in \(P\), if \(pg = f\) then \(g\) factors through \(\overline{f}(Q)\) uniquely via a vertical arrow. That is,
there exists a unique \( g' \) such that \( g = \overline{f}(Q) \circ g' \) and \( pg' = \text{id}_X \).

\[
\xymatrix{
\mathcal{P} & Q \ar[d]_{p} \ar[r]^{f^*Q} & Q \\
\mathcal{C} & X \ar[r]_{f} & Y
}
\]

---

The correspondences \((\_)^*\) and \(\_\) are functorial:

\[
\text{id}_X^* Q = Q , \quad (g \circ f)^* (Q) = f^* (g^* Q) , \quad \overline{g}^*Q = \overline{g}Q \circ \overline{f}(g^*Q) .
\]

The last equality can be depicted as follows.

\[
\xymatrix{
\mathcal{P} & f^*(g^*Q) \ar[d]_{p} \ar[r]^{\overline{f}(g^*Q)} & g^*Q \ar[d]_{\overline{g}^*Q} & Q \\
\mathcal{C} & X \ar[r]_{f} & Y \ar[r]_{g} & Z
}
\]

The category \(\mathcal{P}\) is called the total category of the fibration; \(\mathcal{C}\) is the base category. The arrow \(\overline{f}Q : f^*Q \to Q\) is called the Cartesian lifting of \(f\) and \(Q\). An arrow in \(\mathcal{P}\) is Cartesian (or reindexing) if it coincides with \(fQ\) for some \(f\) and \(Q\).

In the case where \(\mathcal{P}\) is induced by an indexed category \(\Phi : \mathcal{C}^{\text{op}} \to \text{Posets}\) via Def. C.1, a Cartesian lifting is obviously given by \(f^*(Q) = (\Phi f)(Q)\).

In the current paper we focus on poset fibrations (which we shall simply call fibrations). In a (general) fibration a fiber \(\mathcal{P}_X\) is not just a poset but a category, and this elicits a lot of technical subtleties. Nevertheless, it should not be hard to generalize the current paper’s results to general, not necessarily poset, fibrations (especially to the split ones).

We shall often denote a vertical arrow in \(\mathcal{P}\) (i.e. an arrow inside a fiber) by \(\leq\).

C.3. Examples

**Example C.3 (Subobject fibration).** Let \(\mathcal{C}\) be a (well-powered) category with finite limits. The category \(\text{Sub}(\mathcal{C})\) is defined by: its object is a pair \((P, X)\) of \(X \in \mathcal{C}\) and its subobject \(P \hookrightarrow X\) (we write \((P \hookrightarrow X) \in \text{Sub}(\mathcal{C})\)); and its arrow \((P \hookrightarrow X) \xrightarrow{f} (V \hookrightarrow Y)\) is a \(\mathcal{C}\)-arrow \(f : X \to Y\) that restricts to \(P \to Q\). That is, given an arrow \(f : X \to Y\) in \(\mathcal{C}\),

\[
\begin{align*}
\text{\(f\) is an arrow in } \text{Sub}(\mathcal{C}) & \quad \iff \exists f' \text{ s.t. } P \xrightarrow{m} X \xrightarrow{f} Y \xrightarrow{n} Q . \quad (34)
\end{align*}
\]
The projection \((P \rightarrow X) \mapsto X\) defines a functor; thus arises the sub-object fibration \(\downarrow{\text{Sub}(C)}\) of \(C\). In particular, given \(X \xrightarrow{f} Y\) in \(C\) and \((Q \rightarrow Y) \in \text{Sub}(Y)\); the Cartesian lifting \(f^*Q\) is defined by a pullback.

A special case is the following most straightforward modeling of predicate logic. It arises from the contravariant powerset functor \(2(-) : \text{Sets}^{op} \rightarrow \text{Posets}\) via Def. C.1.

**Example C.4** (\(\text{Pred} \downarrow \text{Sets}\)). The subobject fibration \(\downarrow{\text{Sub}(\text{Sets})}\) is denoted by \(\text{Pred} \downarrow \text{Sets}\). An object of its total category is often denoted by \((U \subseteq X)\). Reindexing is given by inverse images.

More concretely, in the category \(\text{Pred}\), an object is a pair \((P,X)\) of a set \(X\) and its subset \(P \subseteq X\); an arrow \((P \subseteq X) \xrightarrow{f} (Q \subseteq Y)\) is a function \(X \xrightarrow{f} Y\) that restricts to \(P \rightarrow Q\) (i.e. \(P \subseteq f^{-1}(Q)\)).

**Example C.5** (Rel). The fibration \(\text{Rel} \downarrow \text{Sets}\) can be introduced from \(\text{Pred} \downarrow \text{Sets}\) via the following change-of-base.

**C.4. Structures in a Fibration**

In a fibration \(\xrightarrow{\mathcal{P}}\), a \(C\)-arrow \(X \xrightarrow{f} Y\) induces a correspondence \(\mathcal{P}_Y \xrightarrow{f^*} \mathcal{P}_X\) via reindexing. This is easily seen to be a monotone map (i.e. a functor between posets as categories).

**Definition C.7** (Fiberwise (co)limits). A fibration \(\xrightarrow{\mathcal{P}}\) is said to have fiberwise limits if:

- each fiber \(\mathcal{P}_X\) has, as a category, all limits (meaning it has arbitrary inf's \(\bigwedge\)); and
- for each \(C\)-arrow \(X \xrightarrow{f} Y\), the reindexing functor \(\mathcal{P}_Y \xrightarrow{f^*} \mathcal{P}_X\) preserves these limits.

In this case each fiber \(\mathcal{P}_X\) has a final object (denoted by \(\top_X\)).

Similarly, a fibration has fiberwise colimits if each fiber has them and they are preserved by reindexing.

The following notions must be distinguished from “fiberwise (co)products.”
**Definition C.8 ((Co)products between fibers).** A fibration $\mathcal{P} \xrightarrow{p} \mathcal{C}$ is said to have products (between fibers) if

- each reindexing functor $f^*: \mathcal{P}_Y \to \mathcal{P}_X$ has a right adjoint $f^* \dashv \prod_f$; and
- the functors $(\prod_f)_f$ satisfy the so-called Beck-Chevalley condition. See (Jacobs, 1999, §1.9).

Similarly, a fibration has coproducts (between fibers) if each reindexing has a left adjoint $\sqcup_f$ and they satisfy the Beck-Chevalley condition.

The prototype example $\text{Pred} \xrightarrow{\rightarrow} \text{Sets}$ has fiberwise (co)limits: each fiber is a complete lattice; and $\wedge$ and $\vee$ are preserved by inverse images. It has (co)products $\sqcup_f$ between fibers, too: specifically $\sqcup_f$ is given by the direct image of the function $f$. See (Jacobs, 1999, §1.9).

Throughout the paper we rely on the following result. It follows from (Jacobs, 1999, Lem. 9.1.2 & Prop. 9.2.1), and extends Lem. 3.5.

**Lemma C.9.** Let $\mathcal{P} \xrightarrow{p} \mathcal{C}$ be a fibration. Assume that $\mathcal{C}$ is complete; then the following are equivalent.

1. The fibration $p$ has fiberwise limits.
2. The total category $\mathcal{P}$ is complete and $p: \mathcal{P} \to \mathcal{C}$ preserves limits.

If this is the case, a limit of a small diagram $(P_I)_{I \in I}$ in $\mathcal{P}$ can be given by

$$\bigwedge_{I \in I}(\pi^*_IP_I) \quad \text{over } \text{Lim}_{I \in I}X_I.$$

Here $X_I := pP_I; \ (\text{Lim}_{I \in I}X_I \xrightarrow{\pi_I} X_I)_{I \in I}$ is a limiting cone in $\mathcal{C}$; and $\bigwedge_{I \in I}$ denotes the limit computed in the fiber $p_{|\text{Lim}_{I \in I}X_I}$.

(Sort of) dually, let $\mathcal{P} \xrightarrow{p} \mathcal{C}$ be a fibration with coproducts $\coprod_f$ between fibers, and assume that $\mathcal{C}$ is cocomplete. Then $p$ has fiberwise colimits if and only if $\mathcal{P}$ is cocomplete and $p: \mathcal{P} \to \mathcal{C}$ preserves colimits. In this case a colimit of a small diagram $(P_I)_{I \in I}$ in $\mathcal{P}$ can be given by

$$\bigvee_{I \in I}(\coprod_{k_f} P_I) \quad \text{over } \text{Colim}_{I \in I}X_I,$$

where $X_I := pP_I$ and $(X_I \xrightarrow{k_f} \text{Colim}_{I \in I}X_I)_{I \in I}$ is a colimiting cocone in $\mathcal{C}$. $\square$

**C.5. Fiberwise Opposite**

Let $\text{op}: \text{Posets} \to \text{Posets}$ be a functor that maps $(\mathcal{P}, \leq)$ to $(\mathcal{P}, \leq)^{\text{op}} = (\mathcal{P}, \geq)$. Assuming a fibration $\mathcal{P} \xrightarrow{p} \mathcal{C}$ is induced—by the Grothendieck construction—by an indexed category $\Phi: \mathcal{C}^{\text{op}} \to \text{Posets}$, the composite $\mathcal{C}^{\text{op}} \xrightarrow{\Phi} \text{Posets} \xrightarrow{\text{op}} \text{Posets}$ induces a fibration in which each fiber is opposed. This is what is denoted by $\mathcal{P}^{\text{op}}\xrightarrow{p^{\text{op}}} \mathcal{C}^{\text{op}}$ in the following lemma.

**Lemma C.10 (Fiberwise opposite, (Bénabou, 1975)).** Let $\mathcal{P} \xrightarrow{p} \mathcal{C}$ be a fibration. There exists a canonical fibration $\mathcal{P}^{\text{op}}\xrightarrow{p^{\text{op}}} \mathcal{C}^{\text{op}}$ such that: $(\mathcal{P}^{\text{op}})_X = (\mathcal{P}_X)^{\text{op}}$; and reindexing functors
coinductive predicates and final sequences in a fibration

coincide, as in the commutative diagram

\[
\begin{align*}
(\mathcal{P}(\text{op}))_Y & \xrightarrow{f^* \text{ in } \mathcal{P}(\text{op})} (\mathcal{P}(\text{op}))_X \\
(\mathcal{P}_Y)^{\text{op}} & \xrightarrow{(f^* \text{ in } p)^{\text{op}}} (\mathcal{P}_X)^{\text{op}}
\end{align*}
\]
for \( f : X \to Y \).

This fibration \( p^{\text{op}} \) is called the fiberwise opposite of \( p \).

**Proof.** We describe the construction of \( \mathcal{P}(\text{op}) \); it is simple in the current setting where we focus on poset fibrations. The objects are the same as those of \( \mathcal{P} \), and the arrows are defined by

\[
\mathcal{P}(\text{op})(P,Q) = \{ f : pP \to pQ \mid P \geq f^* Q \} .
\]

It is easy to show that \( \text{id}_{pP} \) induces an arrow \( P \to P \) in \( \mathcal{P}(\text{op}) \); this gives the identity arrow. The composite \( g \circ f \in \mathcal{P}(\text{op})(P,R) \) of \( f \in \mathcal{P}(\text{op})(P,Q) \) and \( g \in \mathcal{P}(\text{op})(Q,R) \) is given by composition in \( C \), too.

**Appendix D. Omitted Proofs**

D.1. *Proof of Prop. 6.17*

We shall prove the item 1. In the topos \( \text{Sets}^A \), there exists an (Epi, Mono)-factorization, which induces the image functor \( \text{Im}: \text{Sets}^A/P \to \text{Sub}(P) \) that is surjective on objects. In particular, a subpresheaf of \( P \) can be thought of as an image of some arrow with codomain \( P \).

Let \( (Q \xrightarrow{\theta} P) \in \text{Sets}^A/P \). By Lem. 6.15, we may assume \( Q = \text{Colim}_{I \in I} (yB_I) \) for some diagram \( (B_I)_{I \in I} \). We shall show that \( \text{Im} \theta = \bigcup_{I \in I} \text{Im} \theta_I \) where the arrow \( \theta_I \) is the composite \( (yB_I \to \text{Colim}_I (yB_I) \xrightarrow{\theta} P) \).

Firstly, we have the following sublemma.

**Sublemma D.1.** There exists a canonical isomorphism

\[
\left( \text{Colim}_I (yB_I) \xrightarrow{\theta} P \right) \cong \text{Colim}_I (yB_I \xrightarrow{\theta_I} P)
\]

in \( \text{Sets}^A/P \).

**Proof.** Let \( U : \text{Sets}^A/P \to \text{Sets}^A \) be the forgetful functor. The sublemma obviously follows from the facts that comonadic functors create colimits, and that the forgetful functor \( U : \text{Sets}^A/P \to \text{Sets}^A \) is comonadic. Indeed, the comonad for this case is \( (\_ \times P) : \text{Sets}^A \to \text{Sets}^A \) with the counit \( \varepsilon_X = \pi_1 : X \times P \to X \) and the comultiplication \( \delta_X = ((\pi_1, \pi_2), \pi_2) : X \times P \to (X \times P) \times P \). However, we give later a more concrete proof without using the above facts. For reference, we note that the proof via the comonadicity yields the explicit construction below, because any Eilenberg-Moore coalgebra \( (X \xrightarrow{\alpha} X \times P) \) for the comonad \( ((\_ \times P), \varepsilon, \delta) \) is of the form \( \alpha = (\text{id}_X, \xi) \) for some slice \( (X \xrightarrow{\xi} P) \), i.e. a coalgebra \( \xi \) for the constant endofunctor \( P \).

We shall construct the isomorphism (35) by the universality of colimits in \( \text{Sets}^A/P \). Equivalently, we have only to show that there exists a unique fill-in \( \varphi \) for the diagram.
(36) below, which turns out to be an arrow $\varphi : c_1 \to c_2$ between the following cocones over $(yB \xrightarrow{\theta_I} P)_{I \in I}$ in the category $\text{Sets}^A / P$. Let $c_1$ be the cocone induced by the colimiting cocone over $(yB_I)_{I \in I}$ in $\text{Sets}^A$, i.e. the diagram

$$
yB_I \xrightarrow{\kappa_I} \text{Colim}_I(yB_I) \xrightarrow{\varphi} X \xrightarrow{\xi} P
$$

which is natural in $I \in I$. We shall show that the cocone $c_1$ is colimiting, i.e. for an arbitrary cocone $c_2$ there exists a unique arrow $\varphi : c_1 \to c_2$ of cocones. We may define $c_2$ by a diagram

$$
yB_I \xrightarrow{\varphi_I} X \xrightarrow{\xi} P
$$

that is natural in $I \in I$. Consequently, an arrow $\varphi : \text{Colim}_I(yB_I) \to X$ in $\text{Sets}^A$ forms an arrow $\varphi : c_1 \to c_2$ of cocones if and only if for any $I \in I$ the diagram

$$
yB_I \xrightarrow{\kappa_I} \text{Colim}_I(yB_I) \xrightarrow{\varphi} X \xrightarrow{\xi} P
$$

(36)

commutes. The universality of colimits in $\text{Sets}^A$ shows that an arrow $\varphi$ satisfying $\varphi \circ \kappa_I = \varphi_I$ for any $I \in I$ uniquely exists, and that the arrow $\varphi$ with this condition satisfies $\xi \circ \varphi = \theta$.

Secondly in the poset $\text{Sub}(P)$, where colimits are unions, we have

$$\text{Im} \left( \text{Colim}_I(yB_I \xrightarrow{\theta_I} P) \right) = \bigcup_I \text{Im}(yB_I \xrightarrow{\theta_I} P)$$

(37)

because the functor $\text{Im} : \text{Sets}^A / P \to \text{Sub}(P)$ preserves colimits since the functor $\text{Im}$ is left adjoint to the forgetful functor.

Hence we have $\text{Im} \theta = \text{Im}(\text{Colim}_I \theta_I) = \bigcup_I \text{Im} \theta_I$ by (35) and (37). Letting $P = yA$, we obtain $\text{Im}(Q \xrightarrow{\theta} yA) = \bigcup_I \text{Im}(yB_I \xrightarrow{\theta_I} yA)$ for a family $(f_I)_I$ such that $\theta_I = yf_I$; such a family $(f_I)_I$ exists since the functor $y$ is full and faithful. This proves the item 1.

We shall now prove the item 2. We observe that an epi $A \twoheadrightarrow C$ in $A$ induces a mono $yC \hookrightarrow yA$ in $\text{Sets}^A$: this is because the functor $y : A^{\text{op}} \to \text{Sets}^A$ preserves all existing limits, including the pullback

$$
\begin{array}{ccc}
C & \xrightarrow{id_C} & C \\
\downarrow & & \downarrow & m \\
C & \xrightarrow{m} & A
\end{array}
$$

in $A^{\text{op}}$. (The diagram is a pullback if and only if $m$ is a mono in $A^{\text{op}}$, i.e. an epi in $A$.) Thus there is a monotone function $\text{Quot}(A) \to \text{Sub}(yA)$. 
Regarding monos in $A$, we can show the following sublemma (its only-if direction will not be used later).

**Sublemma D.2.** Let $m : C \to B$ be an arrow in $A$. The arrow $ym : yB \to yC$ is an epi in $\text{Sets}^A$ if and only if the arrow $m$ is a split mono.

**Proof.** The following are equivalent (folklore): for an arrow $e$ in $B$,

1. The arrow $e$ is an absolute epi, i.e. $F(e)$ is an epi for any functor $F$ with the domain $B$,
2. the arrow $ye : yB \to yC$ in $\text{Sets}^{B^{\text{op}}}$ is an epi, and
3. the arrow $e$ is a split epi.

The sublemma is part of this fact for $B = A^{\text{op}}$.

To be concrete, we take a retraction $r : B \to C$ of a split mono $m$ in $A$. By $r \circ m = \text{id}_C$, we have $ym \circ yr = \text{id}_yC$, which shows that $ym$ is a (split) epi in $\text{Sets}^A$. Conversely, let $m : C \to B$ be an arrow in $A$ such that $ym : yB \to yC$ is an epi in $\text{Sets}^A$. Because colimits are computed component-wise in the functor category $\text{Sets}^A$, the function $(ym)_C : \text{A}(B, C) \to \text{A}(C, C)$ is surjective. Hence, there exists $r \in \text{A}(B, C)$ such that $(ym)_C(r) = \text{id}_C \in \text{A}(C, C)$, that is, $r \circ m = \text{id}_C$. Therefore the arrow $m$ has a retraction $r$. Therefore an $(\text{Epi}, \text{SplitMono})$-factorization $A \xrightarrow{e} C \xrightarrow{m} B$ in $A$ induces an $(\text{Epi}, \text{Mono})$-factorization $yB \xrightarrow{ym} yC \xrightarrow{ye} yA$. This yields

$$\{ \text{Im}(yB \xrightarrow{yf} yA) \mid B \in A, f : A \to B \} = \{ (yC \xrightarrow{ye} yA) \in \text{Sub}(yA) \mid C \in A, e : A \to C \} \cong \{ (A \xrightarrow{e} C) \in \text{Quot}(A) \mid C \in A \} = \text{Quot}(A),$$

where the last isomorphism holds because the functor $y$ is full and faithful. Hence the item 2 reduces to the item 1.